PROJECTION OF ROOT SYSTEMS

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Abstract

Let \( a \) be a real euclidean vector space of finite dimension and \( \Sigma \) a root system in \( a \) with a basis \( \Delta \). Let \( \Theta \subset \Delta \) and \( M = M_\Theta \) be a standard Levi of a reductive group \( G \) such that \( a_\Theta = a_M/a_G \). Let us denote \( d \) the dimension of \( a_\Theta \), i.e. the cardinal of \( \Delta - \Theta \) and \( \Sigma_\Theta \) the set of all non-trivial projections of roots in \( \Sigma \). We obtain conditions on \( \Theta \) such that \( \Sigma_\Theta \) contains a root system of rank \( d \).

1 Introduction

Let \( a \) be a real euclidean vector space of finite dimension and \( \Sigma \) a root system in \( a \) with a basis \( \Delta \). Let \( \Theta \subset \Delta \), to avoid trivial cases we assume \( \Theta \) is a proper subset of \( \Delta \), i.e. that \( \Theta \) is neither empty nor equal to \( \Sigma \).

We consider \( G \) a quasi-split reductive group over a local field \( F \), and \( T \) a maximal torus of \( G \). As usual, the not-bold notation \( G \) denotes the \( F \)-points of \( G \).

In this article, we fix \( a = a_G^G := a_0/a_G \) quotient of the Lie algebra of the maximal \( F \)-split torus in a maximal torus \( T \) by the Lie algebra of the maximal split torus \( A_M \) in the center of \( G \). We denote \( \Sigma \) the root system of \( G \) and \( \Delta \) a basis of \( \Sigma \). Let \( M \) be a standard Levi subgroup of \( G \) such that the set of simple roots in \( \text{Lie}(M) \) is \( \Delta^M = \Theta \). Then \( a_\Theta = a_M/a_G \).

Let us consider the projection of \( \Sigma \) on \( a_\Theta \) (projection orthogonal to \( \Theta \)) and we denote \( \Sigma_\Theta \) the set of all non-trivial projections of roots in \( \Sigma \). We do not consider the trivial case where \( M = M_0 \) and \( M = G \).

Let us denote \( d \) the dimension of \( a_\Theta \), i.e the cardinal of \( \Delta - \Theta \).

Let us also denote \( \Delta_\Theta \) the set of projections of the simple roots in \( \Delta - \Theta \) on \( a_\Theta \).

In this article, we determine the conditions under which \( \Sigma_\Theta \) contains a root system (for a subspace of \( a_\Theta \)) and what are the types of root system appearing. We will classify the subsystems of rank \( d \) appearing when they exist. We then say they are of maximal rank. Our main results are :

\[ \text{Theorem 1.1. Let } \Sigma \text{ be an irreducible root system of classical type (i.e of type } A, B, C \text{ or } D). \text{ The subsystems in } \Sigma_\Theta \text{ are necessarily of classical type. In addition, if the irreducible (connected) components of } \Theta \text{ of type } A \text{ are all of the same length, and the interval between each of them of length one, then } \Sigma_\Theta \text{ contains an irreducible root system of rank } d \text{ (non necessarily reduced).} \]

\[ \text{Theorem 1.2. Let } \Sigma \text{ be an irreducible root system of exceptional type (i.e of type } E, F_4 \text{ or } G_2). \text{ If } \Sigma_\Theta \text{ contains an irreducible root system of rank } d \text{ it is necessarily of classical type, except in the case of the orthogonal projection to any single root of } E_8 \text{ where } E_7 \text{ appears in the projection.} \]

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1.1 Motivation

This question emerges in an attempt to better understand the result of Silberger in [8]. In Section 3.5 of his work, he claims that

$$\Sigma_\sigma = \{ \alpha \in \Sigma_{\text{red}}(A_{M_1}) | \mu_{(M_1)_\sigma}(\sigma) = 0 \}$$

is a subset of $a_{M_1}^*$ which is a root system in a subspace of $a_{M_1}^*$. Here $\sigma$ is a discrete series representation of a semi-standard Levi subgroup $M_1$ of a reductive group $G$ and $\mu_{(M_1)_\sigma}$ is one factor in the product formula of the $\mu$ function (see also [9], Lemma V.2.1). Let us recall that the $\mu$ function is the main ingredient in the Plancherel measure, the unique Borel measure on the set of irreducible tempered representations of $G$ which was defined by Harish-Chandra to formulate the Plancherel formula for p-adic groups.

Considering the restrictive definition of a root system, it is not immediately clear that the restrictions (resp. projections) of roots to $A_{M_1}$ constitute a root system of rank $|\Delta - \Theta|$, and in general it is not the case. The reader will find many instances of non-existence of maximal rank root system in $\Sigma_\Theta$ in this paper. The goal of this paper is to make precise the conditions on the semi-standard Levi $M_1$ (i.e on $\Theta$) such that $\Sigma(A_{M_1}) := \Sigma_\Theta$ contains a root system of maximal rank.

Silberger’s result applied in the case where $\sigma$ is unitary cuspidal, along with the results obtained in this work are fundamentally used in our work on the Generalized Injectivity Conjecture, a conjecture formulated by Casselman and Shahidi in [2].

**Theorem** (Generalized Injectivity for quasi-split group, [3]). Let $G$ be a quasi-split, connected group defined over a p-adic field $F$ (of characteristic zero). Let $\pi_0$ be the unique irreducible generic subquotient of the standard module $I^G_{\mathbb{R}}(\tau_0)$, let $\sigma$ be an irreducible, generic, cuspidal representation of $M_1$ such that a twist by an unramified real character of $\sigma$ is in the cuspidal support of $\pi_0$.

Suppose that all the irreducible components of $\Sigma_\sigma$ are of type $A, B, C$ or $D$, then, under certain conditions on the Weyl group of $\Sigma_\sigma$, $\pi_0$ embeds as a subrepresentation in the standard module $I^G_{\mathbb{R}}(\tau_\nu)$.

The condition of **maximal rank** of $\Sigma_\sigma$ is also crucial to the existence of a discrete series subquotient in the induced module $I^G_{\mathbb{R}}(\sigma_\lambda)$ whenever $\lambda \in a_{M_1}^*$ is known as a residual point, as studied in [4]. Heiermann’s approach to the infinitesimal character of an irreducible discrete series requires the notion of residual point which itself requires the rank of $\Sigma_\sigma$ to be maximal (see Definition 2.1 in [3]), see also Section 3.8 in [8]. The conditions we have obtained on the form of the Levi $M_1$ in order to obtain a maximal rank root system have already been implicitly used in the literature, see for instance Proposition 1.13 in [5].

Further, understanding which root systems (in particular classical or not) appear in the projections of exceptional root systems helped us circumscribe the limits of our work on the Generalized Injectivity conjecture. One key to understand the limits of our work lies in subtleties involving $W(M_1)$ (the set of representatives in $W$ (Weyl group of $\Sigma$) of elements in the quotient group $\{ w \in W | w^{-1}M_1w = M_1 \} / W^{M_1}$ of minimal length in their right classes modulo $W^{M_1}$) and $W_\sigma$ (Weyl group of $\Sigma_\sigma$). Identifying the potential $\Sigma_\sigma$ of maximal rank in the projections is the first step in doing so.

1.2 Method

Of course, there are always subsystems of rank 1 and as $\Theta$ is assumed to be non-empty there is no need to discuss the case where $\Sigma$ is of rank 2 (in particular $G_2$). We will therefore consider the root systems $\Sigma$ of rank $n \geq 3$ and $d \leq n - 2$. Let us remark that we will find irreducible non reduced root systems: they are the $BC_d$ which contain three subsystems of rank $d: B_d, C_d$ and $D_d$. 


We will use the following remark (see [1, Equation (10) in VI.3, Proposition 12 in VI.4, Chapter VI]).

Let $\alpha$ and $\beta$ be two non-orthogonal distinct elements of a root system. Set

$$C = \left( \frac{1}{\cos(\alpha, \beta)} \right)^2 \quad \text{and} \quad R = \frac{||\alpha||^2}{||\beta||^2}.$$ 

The only possible values for $C$ (the inverse of the square of the cosinus of the angle between two roots) are 4, 2 and $4/3$ whereas assuming the length of $\alpha$ larger or equal to the one of $\beta$, the quotient of the length is respectively 1, 2 or 3. Thus, if $||\alpha|| \geq ||\beta||$

$$\frac{C}{R} \in \{2^2, 1, (2/3)^2\} \quad \text{and} \quad CR = 4.$$ 

We will therefore compute the quotient of lengths and the angles of the non-trivial projections of roots in $\Sigma$, in particular those of elements in $\Delta - \Theta$.

In general $\Sigma_{\Theta}$ is not a root system, however let us observe:

**Lemma 1.3.** The elements in $\Sigma_{\Theta}$ are, in a unique way, linear combination with entire coefficients all with the same sign of the elements in $\Delta_{\Theta}$.

From Theorem 3 (page 156) or Corollary 3 (page 162) in Chapter 6, §1, Sections 6 and 7 in [1]; we know any root in $\Sigma$ can be written in a unique way as linear combination with entire coefficients all with the same sign of the elements in the basis $\Delta$. Then the statement in Lemma 1.3 follows since the projection orthogonal to any subset $\Theta \subset \Delta$ (i.e projection onto $W^\perp$, if $W$ is the vector space generated by $\Theta$) is a linear application.

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2 Classical root systems

In this section, we prove Theorem 1.1 via a case-by-case analysis.

2.1 The case where $\Sigma$ is of type $A_n$ 

Let us consider $a_0$ to be of dimension $n + 1$ and with orthonormal basis $e_1, e_2, \ldots, e_{n+1}$. Let us denote $\Xi$ this ordered basis, i.e the ordered set of the $e_i$. The elements of $\Sigma$ are the $e_i - e_j$ with $i \neq j$; they generate a subspace $a$ of dimension $n$ and $\Delta$ is the set of simple roots $\alpha_i = e_i - e_{i+1}$. Let us denote $\Xi_\Theta$ the projection of $e_i$ on $a_{\Theta}$. The Dynkin diagram of $\Theta$ is a union of irreducible (or connected) components of type $A$. Therefore, the data of $\Theta$ corresponds to a partition of the ordered set $\Xi$ in a disjoint (ordered) union of ordered parts that we index by the smallest index appearing in the indices of the basis vectors associated:

$$\Xi = \Xi_1 \cup \cdots \cup \Xi_l.$$
The correspondence is defined as follows, the part:

\[ \Xi_r = \{e_r, \cdots, e_{r+m}\} \]

is associated to the empty subset if \( m = 0 \) and to the subset of simple roots

\[ \{\alpha_r, \cdots, \alpha_{r+m-1}\} \quad \text{if} \quad m \geq 1. \]

Let us consider an element \( e_i \) in the basis \( \Xi \) of \( a_0 \). Let \( r \) be the smallest integer \( j \) such that \( e_j = e_r \), and let \( r + m \) be the largest. We will have \( e_k = e_r \) for any \( k \) such that \( r \leq k \leq r + m - 1 \).

If \( m = 0 \), it is clear. Observe that if \( m = 0 \), the two simple consecutive roots \( \alpha_i - 1 \) and \( \alpha_i \) where \( e_i \) appears are outside \( \Theta \).

Now, let \( m \geq 0 \), the root \( e_r - e_{r+m} \) has a trivial projection on \( a_0 \) and therefore by Lemma 1.3 all the simple roots that occur in the expression of this root shall be in \( \Theta \). As a result, the roots \( \alpha_k = e_k - e_{k+1} \) belong to \( \Theta \) for any \( k \) such that \( r \leq k \leq r + m - 1 \) and we have:

\[ e_k = e_r + e_{r+1} + \cdots + e_{r+m} \]

for all \( k \) such that \( r \leq k \leq r + m - 1 \). Indeed, this expression of \( e_k \) is then orthogonal to all the roots \( \alpha_k = e_k - e_{k+1} \) for any \( k \) such that \( r \leq k \leq r + m - 1 \).

Such a chain of simple roots is a connected component of length \( m \) of the Dynkin diagram associated to \( \Theta \). We have observed that such a connected component is empty when \( e_r \) is orthogonal to all the elements in \( \Theta \) in which case \( m = 0 \) i.e the two consecutive simple roots \( \alpha_{r-1} \) and \( \alpha_r \) are outside \( \Theta \). If \( e_r \) is associated to a length \( m \) connected component of \( \Theta \) and therefore belongs to an ordered part of cardinal \( m + 1 \) of \( \Xi \), the square of the length of \( \Xi_r \) is:

\[ ||\Xi_r||^2 = \frac{1}{m+1}. \]

Let us consider three vectors \( e_r, e_s \) and \( e_t \) whose projections \( \xi_r, \xi_s \) and \( \xi_t \) are distinct and are associated to three components of \( \Theta \) of type \( A_m, A_p \) and \( A_q \). Let \( \alpha = e_i - e_j \) a root whose projection

\[ \bar{\alpha} = \pm(\xi_r - \xi_s). \]

we obtain

\[ ||\bar{\alpha}||^2 = \frac{1}{m+1} + \frac{1}{p+1}. \]

Let us consider a root \( \beta = e_k - e_l \) whose projection is

\[ \bar{\beta} = \pm(\xi_s - \xi_t) \]

we obtain

\[ ||\bar{\beta}||^2 = \frac{1}{p+1} + \frac{1}{q+1}. \]

and the square of the scalar product of \( \bar{\alpha} \) and \( \bar{\beta} \) is

\[ (\langle \bar{\alpha}, \bar{\beta} \rangle)^2 = \frac{1}{(p+1)^2}. \]

Thus we have:

\[ C = \left( \frac{1}{\cos(\bar{\alpha}, \bar{\beta})} \right)^2 = \left( 1 + \frac{p+1}{m+1} \right) \left( 1 + \frac{p+1}{q+1} \right), \]

and if we assume \( ||\beta|| \geq ||\alpha|| \) i.e \( q \geq m \), we have:

\[ R = \frac{||\alpha||^2}{||\beta||^2} = \frac{1 + \frac{p+1}{m+1}}{1 + \frac{p+1}{q+1}}. \]
Then \( C/R = \left(1 + \frac{p+1}{q+1}\right)^2 \in \{2^2,1,(2/3)^2\} \) and \( CR = \left(1 + \frac{p+1}{m+1}\right)^2 = 4 \).

The only possible case is \( C/R = 4 \) and thus \( R = 1 \) and \( C = 4 \). This implies \( m = p = q \) and \( \{\bar{\alpha},\bar{\beta}\} \) generate a root system of type \( A_2 : \pm(\bar{e}_r-\bar{e}_s), \pm(\bar{e}_s-\bar{e}_t) \) and \( \pm(\bar{e}_r-\bar{e}_t) \).

**Lemma 2.1.** If \( \Sigma \) is of type \( A_n \) the only irreducible subsystems appearing in \( \Sigma_\Theta \) are of type \( A \). To have a root system of rank the dimension \( d \) of \( a_\Theta \) it is necessary if \( d > 1 \), that the Dynkin diagram of \( \Theta \) be a disjoint union of \( d + 1 \) connected components of type \( A_m \) with \( m \geq 0 \), the intervals between each such component being of length one:

\[
n + 1 = (m + 1)(d + 1)
\]

This corresponds to a partition of the ordered basis \( \Xi \) in an union of \( d + 1 \) ordered parts of cardinal \( m + 1 \):

\[
\Xi = \Xi_1 \cup \cdots \cup \Xi_{d+1}
\]

where

\[
\Xi_r = \{e_{(r-1)(m+1)+1}, \ldots, e_{r(m+1)}\}.
\]

In this case \( \Sigma_\Theta \) is of type \( A_4 \).

**Proof.** An irreducible subsystem is necessarily generated by the projections of roots of the form \( \bar{\alpha} = \bar{e}_r - \bar{e}_s \) where the vectors \( \bar{e}_r \) are all of the same length; when we order these vectors following the \( d + 1 \) indices, we obtain a basis of a subspace \( b_0 \) of \( a_0 \) containing a subspace \( b \) of codimension one in which the \( \bar{e}_r - \bar{e}_s \) generate a system of type \( A \). The rest of the corollary follows easily. \( \square \)

### 2.2 The case where \( \Sigma \) is of type \( B_n \)

In this case, the basis of \( a \) is constituted of the \( e_i \) for \( i \in \{1, \ldots, n\} \) and the elements in \( \Sigma \) are the \( \pm e_i \) and the \( \pm e_i \pm e_j \) and \( \Delta \) is formed of the \( \alpha_i = e_i - e_{i+1} \) for \( i \leq n - 1 \) and of \( \alpha_n = e_n \). The set \( \Theta \) is an union of irreducible components which are all of type \( A \) except for at most one which is of type \( B_k \).

We distinguish two cases according to whether \( e_n \) belongs to \( \Theta \) or not, i.e. according to whether one of the components is of type \( B \) or not (case \( k = 0 \)).

Case 1 \( (k = 0) \): \( e_n \notin \Theta \). In this case \( \Theta \) is an union of components of type \( A \). As in the previous case, let us consider three vectors \( e_r, e_s \) and \( e_t \) whose non-trivial projections \( \bar{e}_r, \bar{e}_s \) ans \( \bar{e}_t \) are distinct and associated to three components \( \Theta \) of type \( A_m, A_p \) and \( A_q \). Let us consider the roots of the form \( \alpha = \pm e_i \pm e_j \) and \( \beta = \pm e_k \pm e_l \) and let us suppose their projections write

\[
\alpha = \pm (\bar{e}_r \pm \bar{e}_s) \quad \text{and} \quad \beta = \pm (\bar{e}_r \pm \bar{e}_t) .
\]

The projections are non-trivial, non-collinear, and non-orthogonal. The computations done in the previous subsection show that this family of vectors form a root system if and only if \( m = p = q \). We also have in the projection of \( \Sigma \) the vectors of the form :

\[
\gamma = \pm \bar{e}_v \quad \text{for} \ v \in \{r,s,t\}
\]

Thus a system of type \( B_3 \). Furthermore, \( m \geq 1 \), we also have in the projection of \( \Sigma \), vectors of the form:

\[
\delta = \pm 2\bar{e}_v \quad \text{for} \ v \in \{r,s,t\}
\]
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and in the end we obtain a root system of type $BC_3$.

Let us consider now two roots $\alpha = \pm e_i \pm e_j$ and $\delta = \pm e_k$ whose projections write $\overline{\alpha} = \pm (\epsilon_r \pm \epsilon_s)$ and $\overline{\delta} = \pm \epsilon_r$. We observe that

$$||\overline{\alpha}||^2 = \frac{1}{m+1} + \frac{1}{p+1} \quad \text{and} \quad ||\overline{\delta}||^2 = \frac{1}{p+1}.$$ 

Further $||\overline{\alpha}|| > ||\overline{\delta}||$ and we have:

$$(<\overline{\alpha}, \overline{\delta}>)^2 = \frac{1}{(p+1)^2}.$$ 

Therefore

$$C = \left(\frac{1}{\cos(\overline{\alpha}, \overline{\delta})}\right)^2 = 1 + \frac{p + 1}{m + 1} \quad \text{and} \quad R = \frac{||\overline{\alpha}||^2}{||\overline{\delta}||^2} = 1 + \frac{p + 1}{m + 1}$$

So we have $C = R$ which forces $C = R = 2$ and we recover the condition $m = p$.

Let us also remark that two short roots (that is of type $\pm \epsilon_r$) or long (that is of type $\pm 2\epsilon_r$) (the length being relative to the length of roots $\pm (\epsilon_s \pm \epsilon_t)$) are necessarily proportional or orthogonal. This observation excludes the occurrence of a root system of type $F_4$. Combining these observations, we see that except if $m = 0$ (trivial case where the projection is the identity), we obtain maximal subsystems of type $BC$ (in particular non reduced).

Case 2 ($k \geq 1$): $e_n \in \Theta$. The projection on the orthogonal complement of $\epsilon_n$ gives a system $B_{n-1}$ and reiterating this procedure when $\Theta$ contains $B_k$, we recover the case 1 previously treated for $B_{n-k}$.

In conclusion, we have proven:

**Lemma 2.2.** The maximal subsystems are of type $B$ or $BC$. These contain the subsystems of type $B$, $C$ or $D$ of the same rank. Let us assume $e_n$ belongs to a connected component of length $k$ (then of type $B_k$), with $k \geq 0$ (the case $k = 0$ is the case in which $e_n$ does not belong to $\Theta$). Then, the set $\Sigma_{\Theta}$ contains a system of rank equal to the dimension $d$ of $a_\Theta$ if the other components are all of the same length $m$ (and type $A_m$), the intervals between any of these components being of length one with $n - k = (m + 1)d$.

The projected system is of type $BC_d$ except if $m = 0$ in which case we obtain $B_{n-k}$.

**Case 1** : $k = 0$, $n = d(m + 1)$ : the projected system is of type $BC_d$ if $m \geq 1$.

**Case 2** : $k \geq 1$, $n - k = d(m + 1)$ : the projected system is of type $BC_d$.

This corresponds to a partition of the ordered basis $\Xi$ of cardinal $n$ in a union of $d + 1$ ordered parts

$$\Xi = \Xi_1 \cup \cdots \cup \Xi_{d+1}$$

where

$$\Xi_r = \{e_{(r-1)(m+1) + 1} \cdots e_{r(m+1)}\} \quad \text{for} \; 1 \leq r \leq d \; \text{and} \; \Xi_{d+1} = \{e_{d(m+1)+1} \cdots e_{d(m+1)+r}\}.$$

**2.2.1 The case where $\Sigma$ is of type $C_n$**

In this case the basis of $a$ is formed with the $e_i$ for $i \in \{1, \cdots, n\}$ and the elements of $\Sigma$ are the $\pm 2e_i$ and the $\pm e_i \pm e_j$ ; moreover $\Delta$ is constituted of the $\alpha_i = e_i - e_{i+1}$ for $i \leq n - 1$ and of $\alpha_n = 2e_n$. The set
\( \Theta \) is an union of irreducible components all of type \( A \) except for at most one of type \( C_k \). We distinguish two cases whether \( e_n \) belongs or not to \( \Theta \).

Case 1 \((k = 0)\) : \( 2e_n \notin \Theta \). In this case \( \Theta \) is an union of components of type \( A \). As in the case of \( \Sigma \) of type \( A_n \), let us consider three vectors \( e_r, e_s \) and \( e_t \) whose projections (which are non-zero) \( \overline{e_r}, \overline{e_s} \) et \( \overline{e_t} \) are distinct and associated to three components of \( \Theta \) of type \( A_m, A_p \) and \( A_q \) and roots \( \alpha = \pm e_i \pm e_j \) and \( \beta = \pm e_k \pm e_t \) whose projections are

\[
\overline{\alpha} = \pm (\overline{e_r} \pm \overline{e_s}) \quad \text{and} \quad \overline{\beta} = \pm (\overline{e_s} \pm \overline{e_t})
\]

They will constitute a root system if and only if \( m = p = q \). Then we obtain a root system of type \( C_3 \) constituted of the \( \pm (\overline{e_r} \pm \overline{e_s}), \pm (\overline{e_s} \pm \overline{e_t}), \pm (\overline{e_r} \pm \overline{e_t}) \) and \( \pm 2\overline{e_v} \) for \( v \in \{r, s, t\} \).

Let us now consider the two roots \( \alpha = \pm e_i \pm e_j \) and \( \beta = \pm 2e_k \) whose projections write \( \overline{\alpha} = \pm (\overline{e_r} \pm \overline{e_s}) \) and \( \overline{\beta} = \pm 2\overline{e_s} \).

\[
|\overline{\alpha}|^2 = \frac{1}{m+1} + \frac{1}{p+1} \quad \text{and} \quad |\overline{\beta}|^2 = \frac{4}{p+1}
\]

and therefore

\[
\left( \langle \overline{\alpha}, \overline{\beta} \rangle \right)^2 = \frac{4}{(p+1)^2} \quad \text{and} \quad C = \left( \frac{1}{\cos \langle \overline{\alpha}, \overline{\beta} \rangle} \right)^2 = \left( 1 + \frac{p+1}{m+1} \right).
\]

If we assume \( |\overline{\beta}| \geq |\overline{\alpha}| \) we have

\[
R = \frac{|\overline{\beta}|^2}{|\overline{\alpha}|^2} = \frac{4}{(1 + \frac{p+1}{m+1})}
\]

and \( CR = 4 \). All the cases are a priori possible.

If \( C = 2 \) et \( R = 2 \) then we necessarily have \( p = m \). The vectors \( \overline{\alpha} \) and \( \overline{\beta} \) are the basis of a root system of a type \( C_2 \) where \( \overline{\beta} \) is the long root. The roots are

\[
\pm \overline{\alpha} = \pm (\overline{e_r} - \overline{e_s}) \quad , \quad \pm \overline{\beta} = \pm 2\overline{e_s} \quad , \quad \pm (\overline{\alpha} + \overline{\beta}) = \pm (\overline{e_r} + \overline{e_s}) \quad \text{and} \quad \pm (2\overline{\alpha} + \overline{\beta}) = \pm 2\overline{e_r}.
\]

The case \( C = 4 \) and \( R = 1 \) implies

\[
(p+1) = 3(m+1) \quad \text{and therefore} \quad p = 3m + 2.
\]

Then \( |\overline{\alpha}| \) and \( |\overline{\beta}| \) constitute the basis of a a root system of type \( A_2 \) whose roots are

\[
\pm \overline{\alpha} = \pm (\overline{e_r} - \overline{e_s}) \quad , \quad \pm \overline{\beta} = \pm 2\overline{e_s} \quad \text{and} \quad \pm (\overline{\alpha} + \overline{\beta}) = \pm (\overline{e_r} + \overline{e_s})
\]

but the vector \( \pm 2\overline{e_r} \) does not contribute to this system.

Finally if \( C = 4/3 \) we have

\[
3(p+1) = (m+1) \quad \text{and therefore} \quad m = 3p + 2.
\]

This forces \( R = 3 \) which is a configuration of simple roots for a root system of type \( G_2 \) where \( \overline{\beta} \) is the long root. However, \( \Sigma_{|\phi|} \) does not contain all the necessary roots for such a system ; indeed the root

\[
\overline{\beta} + 3\overline{\alpha} = 3\overline{e_r} - \overline{e_s}
\]

is not obtained.

Let us assume \( |\overline{\alpha}| \geq |\overline{\beta}| \) we have \( C/R = 4 \) and we recover the case \( C = 4, R = 1 \) and therefore \((p+1) = 3(m+1)\).
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Case 2 \((k \geq 1)\) : \(e_n \in \Theta\). The projection on the orthogonal complement of \(e_n\) gives a system of type \(BC_{n-1}\). And, reiterating this procedure, we recover the case of \(BC_{n-k}\) which can be treated using our previous considerations on \(B_{n-k}\) and \(C_{n-k}\).

To conclude, we have proved:

**Lemma 2.3.** The maximal subsystems are of type \(A, B, C, D\). Let us assume \(2e_n\) belongs to a connected component of length \(k\) (and type \(C_k\)), with \(k \geq 1\). The projection on the orthogonal of this component is a root system of type \(BC_{n-k}\). We recover the case where \(k = 0\), i.e. where \(e_n\) does not belong to \(\Theta\) for a system of type \(BC\).

If \(d \geq 3\) the set \(\Sigma_\Theta\) contains a system of rank equal to the dimension \(d\) of \(a_\Theta\) if the other components are all of the same length \(n \geq 0\) (and type \(A_m\)), the intervals between any of these components being of length one with \(n - k = (m + 1)d\), then the projected system is of type \(BC_d\) (or \(C_n\) if \(k = 0\) and \(m = 0\), trivial case excluded).

If \(d = 2\) we obtain either \(BC_d\) when the two components of type \(A\) are of length \(m\) or \(A_2\) when \((p + 1) = 3(m + 1)\).

The case \(k = 0\), with \(n = (m + 1)d\) and projected system \(C_d\)

\[
\begin{array}{cccccccc}
& & & & & & & \\
A_m & & A_m & & A_m & & & \\
& & & & & & & \\
\end{array}
\]

The case \(k = 0\), with \(p = 3m + 2\) and \(n = 4(m + 1)\), and projected system containing \(A_2\)

\[
\begin{array}{cccccccc}
& & & & & & & \\
A_m & & A_p & & & & & \\
& & & & & & & \\
\end{array}
\]

The case \(k \geq 1\), with \(n - k = (m + 1)d\) and projected system \(BC_d\)

\[
\begin{array}{cccccccc}
& & & & & & & \\
A_m & & A_m & & A_m & & C_k & \\
& & & & & & & \\
\end{array}
\]

The case \(k \geq 1\), with \(p = 3m + 2\) and \(n - k = 4(m + 1)\), the projected system contains \(A_2\)

\[
\begin{array}{cccccccc}
& & & & & & & \\
A_m & & A_p & & C_k & & & \\
& & & & & & & \\
\end{array}
\]

2.2.2 The case where \(\Sigma\) is of type \(D_n\)

With the notations analogous to the previous cases the roots are the \(\pm e_i \pm e_j\) and \(\Delta\) is constituted of \(\alpha_i = e_i - e_{i+1}\) for \(i \leq n - 1\) and of \(\alpha_n = e_{n-1} + e_n\).

Case 1 : \(\alpha_{n-1} = e_{n-1} - e_n\) and \(\alpha_n = e_{n-1} + e_n\) are in \(\Theta\) and the orthogonal complement of \(\Theta\) admits the \(e_i\) for \(1 \leq i \leq n - 2\) as a basis. The projection on the orthogonal of \(e_n\) and \(e_{n-1}\) contain the \(\pm e_i \pm e_j\) along with the roots \(\pm e_i\) for \(i\) and \(j\) between 1 and \(n - 2\) obtained projecting the \(\pm(e_i - e_n)\). We, therefore, obtain the system \(B_{n-2}\) already considered above.

Case 2 : \(\alpha_{n-1} = e_{n-1} - e_n\) is in \(\Theta\) but \(e_{n-1} + e_n\) is not. As in the case of root system of type \(B_n\) let us consider the three vectors \(e_r, e_s\) and \(e_t\) whose non-zero projections \(e_r, e_s\) and \(e_t\) are distinct and associated to three components of \(\Theta\) of type \(A_m, A_p\) and \(A_q\). Once projected we find the \(\pm e_r \pm e_s\) and \(\pm e_r \pm e_t\). We also have

\[
2e_r = e_r + e_{r+1} = 2e_{r+1}
\]

if \(e_r = e_r - e_{r+1}\) belongs to a connected component of \(\Theta\). Therefore \(\Sigma_\Theta\) contains a root system of type \(C_d\) if all the connected components of \(\Theta\) are of the same cardinal \(m\) with \(n = d(m + 1)\).
Case 2' : analogous to the case 2 when exchanging $e_n$ with $-e_n$.

Case 3 : Neither $\alpha_{n-1} = e_{n-1} - e_n$ nor $\alpha_n = e_{n-1} + e_n$ are in $\Theta$.

We, therefore, have either an analogous situation to the one treated for $A_n$, or we consider $\alpha = \pm e_n - 1 \pm e_n$ and $\beta = e_n \pm e_{n-1}$.

In this case we have $e_n = e_n$ and therefore with the now familiar notations

$$R = \frac{(1 + (p + 1))}{(1 + \frac{p+1}{m+1})} \quad \text{and} \quad C = (1 + (p + 1)) \left(1 + \frac{p+1}{m+1}\right)$$

Therefore

$$\frac{C}{R} = (1 + \frac{p+1}{m+1})^2$$

which forces $R = 1$ and $C = 4$; thus $p = m = 0$. The existence of a system of maximal rank in the projection for a configuration of this sort forces $m_i = 0$ for any $i$, that is $\Theta$ is empty, a case which is possible but trivial hence excluded a priori.

To sum up, we have proven the :

**Lemma 2.4.** For a system of type $D$ the subsystems in the projection are of type $A$, $B$, $C$ or $D$. If $\alpha_{n-1} = e_{n-1} - e_n$ and $\alpha_n = e_{n-1} + e_n$ are in $\Theta$ and if the others components of $\Theta$ are all of type $A_m$, the interval between two such components are of length one, with $n - k = (m+1)d$, then there exists a system of type $BC_d$ in $\Sigma_{\Theta}$. In the case 2 or 2', the projection contains a system of maximal rank of type $C_d$ if all the components are of type $A_m$ and if $n = (m+1)d$.

The case 1 : $D_k \subset \Theta$ with $k \geq 2$ ; we recover the case of $B_{n-k}$.

The case 2 (or 2') : The projection contains a rank maximal system of type $C_d$ if all the components are of type $A_m$ and if $n = (m+1)d$.

### 3 Exceptional root systems

As opposed to the previous treatment in the context of classical root systems, the case of exceptional groups requires a cumbersome case-by-case analysis, which leads to the following result :

**Theorem (1.2).** Let $\Sigma$ be an irreducible root system of exceptional type (i.e of type $E$, $F_4$ or $G_2$). If $\Sigma_{\Theta}$ contains an irreducible system of rank $d$ it is necessarily of classical type, except in the case of the orthogonal projection to any single root of the roots of $E_8$ where $E_7$ appears in the projection.

As a result of this case-by-case analysis, we also give the most exhaustive description of subsystems of $\Sigma_{\Theta}$ of rank greater or equal to 2. In most of those cases, we exhibit a basis for the root system of largest rank obtained. We have verified for each case that those subsystems were indeed of largest rank in the projection although we have not written systematically all justifications.
Remark 3.1. Let us explain here two important observations made in the case of exceptional root systems.

1. In almost all cases, in order to obtain a subsystem \( S \) of \( \Sigma_\Theta \) of rank \( d \), one has to consider a basis \( \Delta_S \) constituted of at least some projections of non-simple roots. This observation contrasts with classical root systems, where as the reader has noticed, \( \Delta_S \) is constituted of projections of simple roots except possibly for the last root of \( \Delta_S \), i.e. the one on the extreme right of the Dynkin diagram constituted from those simple roots.

2. In the root systems of type \( E \), the only root systems of rank \( d \) appearing in the projection are of type \( A \) or \( D \).

The results of Theorem 1.2 rely on the three following Lemmas:

Lemma 3.2. Let \( \Sigma \) be an irreducible root system. Any two subsystems \( \Theta \) and \( \Theta' \) with only one root of \( \Sigma \) of the same length are conjugated under the Weyl group \( W \). If \( w \) is a Weyl group element sending \( \Theta \) on \( \Theta' \), we have \( \Sigma_{\Theta'} = w(\Sigma_{\Theta}) \).

Proof. First by an equivalent of the incomplete basis theorem for root systems (see [1], Chap VI, 1, Proposition 24), it is always possible to complete any root to a basis of the root system. Hence it is enough to consider the case of simple roots.

By a classical lemma (see for instance Lemma C in [6], III, 10), all roots of the same length are conjugate under \( W \). Let us consider any two basis’ roots of the same length, \( \alpha \), and \( \beta = w(\alpha) \). Let us assume we project all vectors in \( \Sigma \) orthogonally to \( \Theta = \{ \alpha \} \), and \( \Theta' = \{ \beta \} \). Since the Weyl group is a subgroup of the isometry group of the root system, it preserves lengths and angles. Therefore applying first \( w \in W \) to all roots and projecting with respect to \( \beta \) yields the same result as applying \( w \in W \) to \( \Sigma_{\Theta} \).

Remark 3.3. Under the conditions of Lemma 3.2, if \( \Sigma_{\Theta} \) contains a maximal root system of rank \( d \), then it doesn’t depend on the choice of the root generating \( \Theta \); It is enough to determine \( \Sigma_{\Theta} \) for any root in \( \Sigma \). Indeed the ratios and angles between roots in the maximal root system occurring in \( \Sigma_{\Theta} \) and \( w(\Sigma_{\Theta}) \) are the same.

Lemma 3.4. Let \( \Sigma \) be the root system of an exceptional group. No root system of type \( G_2 \) appears as a subsystem of rank \( d \) in \( \Sigma_{\Theta} \).

Proof. The conditions to obtain \( G_2 \) as a subsystem of rank 2 in \( \Sigma_{\Theta} \) are:

1. The cardinal of \( \Delta - \Theta \) to be equal to two.

2. Considering the projections \( \alpha \) and \( \beta \) of two roots in \( \Sigma \), the values of \( C = 4/3 \) and \( R = 3 \).

These conditions are verified in the case \( E_6, \Theta = \{ \alpha_2, \alpha_3 \} \cup \{ \alpha_5, \alpha_6 \} \) studied in Lemma 3.8. The squared norm equals to \( 6/9 \) is specific to \( E_6 \) and will not appear in the case \( E_7, \Theta = \{ \alpha_2, \alpha_3 \} \cup \{ \alpha_5, \alpha_6, \alpha_7 \} \) neither in the case \( E_8, \Theta = \{ \alpha_2, \alpha_3 \} \cup \{ \alpha_5, \alpha_6, \alpha_7, \alpha_8 \} \). It might be possible to obtain one root of squared norm containing a factor 3 (for example, in \( E_8, \Theta = \{ \alpha_2, \alpha_3 \} \cup \{ \alpha_1, \alpha_6, \alpha_7, \alpha_8 \} \), the root \( e_3 + e_4 \) has squared norm equal to \( 4/3 \)). The case to consider are when there is two consecutive roots in the Levi (such as \( \alpha_2 \) and \( \alpha_3 \)) completed by others which are not their immediate neighbours (second branch), but the number of roots in the second branch always lead to inappropriate factors \( R \) and \( C \). This observation along with the results in \( F_4 \) presented in the Subsection 3.1 yield the result.

Lemma 3.5. Let \( \Sigma \) be the root system of an exceptional group. No root system of type \( F_4 \) appears as a subsystem of rank \( d \) in \( \Sigma_{\Theta} \).
Projection of root systems

Proof. We are looking for ratio $R = 2$ and $C = 2$ and in particular for roots with squared norms equal to 1. We consider here in particular the projections of roots of the form $\frac{1}{2}|±e_0 ± e_1 ± e_2 ⋯ ± e_6 ± e_7|$ to complete the details given in the case by case analysis below.

In $E_6$, if $Θ = \{α_1, α_i\}$, the projections of roots of the form $\frac{1}{2}[e_0 ± e_1 ± e_2 ⋯ ± e_6 ± e_7]$ give only roots of squared norms $3/2$ or 2. In case $α_1 \notin Θ = \{α_k, α_l\}$, some roots of the form $\frac{1}{2}[e_0 − e_{i,1} + e_{i,2} − e_7]$ or $\frac{1}{2}[e_0 + e_{i,1} − e_{i,2} − e_7]$ (call a root of this sort $β$) whose squared norms equal one appear in the projection. There are also some roots of the form $\frac{1}{2}[e_0 ± e_1 ± e_2 ⋯ ± e_6 − e_7]$ (call a root of this sort $α$) whose squared norms equal two. Considering the scalar product between $α$ and $β$: they are either orthogonal, or $C = 2$ and $R = 2$. The issue is that in the $F_4$ basis, one needs two roots of norm 1 whose scalar product is $-1/2$; and their sum needs to appear in the projection. Here making variations on $±\frac{1}{2}[e_0 − e_{i,1} + e_{i,2} − e_7]$ is the only option to have roots of norm 1, and they don’t satisfy the latter conditions.

In $E_7$, one considers the roots in $Θ$ to be “two consecutive plus one” (such as $Θ = \{α_2, α_3, α_7\}$ or $Θ = \{α_2, α_5, α_6\}$) and then roots of the form $\frac{1}{2}[±e_0 ± e_1 ± e_2 ⋯ ± e_6 ± e_7]$ have norms 2 or 3/2. Or one considers three consecutive roots (such as $Θ = \{α_5, α_6, α_7\}$) where roots of norms 1 of the form the

$$\frac{1}{2}[±e_i ± e_j ± e_k ± e_l]$$

(for instance $\frac{1}{2}[±e_0 ± e_1 ± e_2 ± e_3]$) appear in the projections. Obviously, since they have to be orthogonal to all the roots in $Θ$ and are projections of roots in $E_7$ (constraints on the number of negative signs), as opposed to the roots of $F_4$ of this form, not all of the $2^4$ roots of this form are obtained!

In $E_8$, the case of $Θ = \{α_4, α_5, α_6, α_7\}$ (resp. $Θ = \{α_1, α_3, α_5, α_8\}$) yields roots of norms 1 among the $\frac{1}{2}[±e_0 ± e_1 ± e_2 ± e_3]$ (resp. $\frac{1}{2}[±e_0 ± e_7 ± e_1 ± e_6]$). Again, since they have to be orthogonal to all the roots in $Θ$ and are projections of roots in $E_8$ (constraints on the number of negative signs), as opposed to the roots of $F_4$ of this form, not all of the $2^4$ roots of this form are obtained! Furthermore, as there are roots of the form $e_i − e_j$ but no root of the form $e_i$; obtaining the 48 roots of an $F_4$ root system is not possible.

The following lemma will be used to study various cases below.

Lemma 3.6. If $\frac{α_i}{α_i}$ is $e_i − \frac{e_{i−1} + e_{i+1}}{2}$ (resp. $\frac{α_i}{α_i} = \frac{e_{i−1} + e_{i+1}}{2} − e_{i−1}$), the roots $\{α_i, α_{i+1} = α_{i+1}\}$ (resp. $\{α_i, α_{i−1} = α_{i−1}\}$) cannot be the simple roots of a root system in $Σ_Θ$.

Proof. The squared length of $\frac{α_i}{α_i}$ is $3/2$. The squared length of $\frac{α_{i−1}}{α_{i−1}}$ (resp. $\frac{α_{i+1}}{α_{i+1}}$) is 2. The ratio is $4/3$. $\frac{1}{\sqrt{3/2}} = 1/3$ therefore $C = 3$. This is not a valid value for $C$ to obtain a root system of rank 2.

3.1 The case $F_4$

In this case $a$ has for basis the $e_i$ for $i \in \{1, 2, 3, 4\}$ and the elements of $Σ$ are the $±e_i$, the $±e_i ± e_j$ (i $\neq j$) and the $1/2(±e_1 ± e_2 ± e_3 ± e_4)$. Furthermore, $Δ$ is of the form: $α_1 = e_1 − e_2$, $α_2 = e_2 − e_3$, $α_3 = e_3$ and $α_4 = −1/2(e_1 + e_2 + e_3 + e_4)$. There are ten $Θ$ with 1 or 2 elements, we examine each case separately.

Case 1 : $Θ = \{α_1\}$
where

\[ \alpha_4 = \alpha_4 \]
\[ \alpha_3 = \alpha_3 \]
\[ \alpha_2 = \frac{e_1 + e_2}{2} - e_3 \]

The squared norms of \( \alpha_4 \) and \( \alpha_3 \) are 1, whereas the squared norm of \( \alpha_2 \) is 3/2. The roots \( \alpha_3, \alpha_4, \) and \( e_1 + e_2 \) form the basis of a root system of type \( C_3 \).

Case 2: \( \Theta = \{ \alpha_2 \} \)

The squared norms of \( \alpha_1 \) and \( \alpha_3 \) are respectively 3/2 and 1/2, whereas the squared norm of \( \alpha_4 \) is 1. The roots \( e_1, \alpha_4, \) and \( e_2 + e_3 \) form the basis of a root system of type \( C_3 \).

Case 3: \( \Theta = \{ \alpha_3 \} \)

We observe that \( \Sigma_\Theta \) contains \( B_3 \) with the \( \pm e_i \), and the \( \pm e_i \pm e_j \) \( (i \neq j) \) for \( i \) and \( j \) in \( \{1, 2, 4\} \) as maximal rank subsystem. The projection of \( \alpha_4 \) and the analogous roots \( 1/2(\pm e_1 \pm e_2 \pm e_4) \) do not contribute to a highest rank subsystem.

Case 4: \( \Theta = \{ \alpha_4 \} \)

We have

\[ e_i = e_i - \frac{(e_1 + e_2 + e_3 + e_4)}{4} \]

and the projection of \( \Delta - \Theta \) is made of

\[ \alpha_1 = e_1 - e_2 \]
\[ \alpha_2 = e_2 - e_3 \]
\[ \alpha_3 = e_3 - \frac{(e_1 + e_2 + e_3 + e_4)}{4} = \frac{(3e_3 - e_1 - e_2 - 4e_4)}{4} \]

whose squared lengths are respectively 2, 2 and 12/16=3/4.

The \( \pm(e_i - e_j) \) with \( i \neq j \) in \( \{1, 2, 3\} \) constitute a root system of type \( A_2 \).

Since the root \( \frac{(e_1 - e_2 + e_3 - e_4)}{2} \) is orthogonal to \( \alpha_4 \), it appears in the projection, together with \( e_2 - e_3 \) they form the basis of a root system \( B_2 \) appearing in the projection. Further, \( \{e_3 - e_1, e_2 - e_3, \frac{(e_1 - e_2 + e_3 - e_4)}{2}\} \) constitute the basis of a root system of type \( B_3 \), and all the sums of this basis’ roots occur in the projection. Therefore \( B_3 \) is therefore of highest rank in the projection.

Case 5: \( \Theta = \{ \alpha_3, \alpha_4 \} \)
Projection of root systems

The projection of $\Delta - \Theta$ is:

$$\alpha_1 = \alpha_1 = e_1 - e_2 \quad \text{and} \quad \alpha_2 = e_2 - \frac{e_1 + e_2 + e_4}{3} = \frac{2e_2 - e_1 - e_4}{3}$$

whose squared lengths are respectively $2$ and $6/9=2/3$. Therefore $R = 3$ and $C = 4/3$, which is compatible with a root system of type $G_2$. However, $\alpha_1 + \alpha_2 = \frac{2e_1 - e_2 - e_4}{3}$ is not the projection of a root in $\Sigma$.

Case 6 : $\Theta = \{\alpha_1, \alpha_2\}$

\[ \begin{array}{c}
\bullet  \\
\circ \\
\circ \\
\circ
\end{array} \]

The projection of $\Delta - \Theta$ is:

$$\alpha_3 = \frac{e_1 + e_2 + e_3}{3} \quad \text{and} \quad \alpha_4 = \alpha_4$$

whose squared lengths are respectively $1/3$ and $1$.

$C=4/3$, $R=3$, these are the conditions for a configuration of type $G_2$.

However, we notice that $\alpha_1 + \alpha_3$ and $\alpha_4 + 2\alpha_3$ do not occur in the projection.

Case 7 : $\Theta = \{\alpha_1, \alpha_4\}$

\[ \begin{array}{c}
\bullet  \\
\circ \\
\bullet \\
\circ
\end{array} \]

The projection of $\Delta - \Theta$ is:

$$\alpha_3 = e_3 - \frac{e_1 + e_2 + e_3 + e_4}{4} = \frac{(-e_1 - e_2 - e_4 + 3e_3)}{4} \quad \text{and} \quad \alpha_2 = \frac{e_1 + e_2}{2} - e_3$$

whose squared lengths are respectively $3/2$ and $3/4$. The value of $C$ is $9/8$. No root system of rank 2 satisfies this condition. Looking at projections of non-simple roots does not yield any further potential basis of root system in the projection.

Case 8 : $\Theta = \{\alpha_2, \alpha_3\}$

\[ \begin{array}{c}
\circ  \\
\bullet \\
\bullet \\
\bullet
\end{array} \]

The projection of $\Delta - \Theta$ is:

$$\alpha_1 = \frac{-e_1 - e_4}{2} \quad \text{and} \quad \alpha_1 = e_1$$

whose squared lengths are respectively $1/2$ and $1$. The ratio of squared lengths is 2, and the squared scalar product is $1/4$, therefore $C = 2$. We observe $\Sigma_\Theta$ contains $B_2$ with the $\pm e_i$, and the $\pm e_i \pm e_j$ ($i \neq j$) for $i$ and $j$ in $\{1, 4\}$ as highest rank subsystem.

Case 9 : $\Theta = \{\alpha_2, \alpha_4\}$

\[ \begin{array}{c}
\bullet  \\
\circ \\
\bullet \\
\circ
\end{array} \]

$$\alpha_1 = e_1 - \frac{e_2 + e_3}{2}$$

whose squared length is $3/2$,

$$\alpha_3 = \frac{e_2 + e_3}{2} - \frac{e_1 + e_2 + e_3 + e_4}{4}$$

whose squared length is $1/2 + 1/4 = 3/4$,

$$1/C = \frac{1/4}{3/2 \times 3/4} = 2/9, \quad \text{hence} \quad C = 9/2$$

Looking at projections of non-simple roots does not yield any further potential basis of root system in the projection. No root system can be obtained in $\Sigma_\Theta$.
Case 10 : $\Theta = \{ \alpha_1, \alpha_3 \}$

The squared length of $\alpha_4$ is $3/4$. We have $R = 3/2$ and $1/C = \frac{1/4}{1/2 \times 3/4} = \frac{2}{3}$. Therefore $C = 3/2$ and there is no root system satisfying such condition.

The root $e_1 + e_2$ of squared norm 2 also appears in the projection but the values of $C$ obtained while considering it together with $\alpha_1$ or $\alpha_2$ is incompatible with any root system. No root system can be obtained in $\Sigma_{\Theta}$.

### Table 1: Roots system occurring in $\Sigma_{\Theta}$ for $\Sigma$ of type $F_4$

<table>
<thead>
<tr>
<th>$\Theta$ = {...}</th>
<th>squared lengths of projected roots</th>
<th>chosen roots to calculate $C$ and $R$</th>
<th>$C$ and $R$</th>
<th>root system of highest rank obtained (of rank $\geq 2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_1$</td>
<td>3/2 and 1</td>
<td></td>
<td>$C_3$</td>
<td></td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>3/2, 1/2, 1</td>
<td></td>
<td>$B_3$</td>
<td></td>
</tr>
<tr>
<td>$\alpha_3$</td>
<td></td>
<td></td>
<td>$B_3$</td>
<td></td>
</tr>
<tr>
<td>$\alpha_4$</td>
<td>2 and 3/4</td>
<td>$\alpha_1, \alpha_2$</td>
<td>$B_3$</td>
<td></td>
</tr>
<tr>
<td>$\alpha_3, \alpha_4$</td>
<td>2 and 2/3</td>
<td>$\alpha_1, \alpha_2$</td>
<td>None</td>
<td></td>
</tr>
<tr>
<td>$\alpha_1, \alpha_2$</td>
<td>1/3, 1</td>
<td>$\alpha_1, \alpha_2$</td>
<td>4/3 and 3</td>
<td>None</td>
</tr>
<tr>
<td>$\alpha_1, \alpha_4$</td>
<td>$\alpha_4$ has squared length 3/2, $\alpha_2$ 3/4</td>
<td>$\alpha_2, \alpha_4$</td>
<td>$C=9/8$</td>
<td>None</td>
</tr>
<tr>
<td>$\alpha_2, \alpha_3$</td>
<td>1 and 1/2</td>
<td>$\alpha_1, \alpha_4$</td>
<td>$C=2$</td>
<td>$B_2$</td>
</tr>
<tr>
<td>$\alpha_2, \alpha_4$</td>
<td>3/2 and 3/4</td>
<td>$\alpha_1, \alpha_4$</td>
<td>$C=9/2$</td>
<td>None</td>
</tr>
<tr>
<td>$\alpha_1, \alpha_3$</td>
<td>1 and 3/4</td>
<td></td>
<td>$R=3/2, C=3/2$</td>
<td>None</td>
</tr>
</tbody>
</table>

### 3.2 The case of root systems of type $E$

We will use the $E$ basis as proposed by Jean-Pierre Labesse (see [7] and an unpublished note): the details are given below (see also the remark 3.7). We say roots are of type $A$ when they are of the form $\pm (e_i - e_j)$, of type $D$ when they are of the form $\pm (e_i + e_j)$ and of type $E$ when they are of the form $\frac{1}{2} [\pm e_0 \pm e_1 \pm e_2 \cdots \pm e_6 \pm e_7]$ with an even number of signs (and some further conditions in $E_7$ and $E_8$).

#### 3.2.1 The occurrence of roots of type $E$ in the projection

We need the expression $\alpha_1 = \frac{(e_0 - e_2) + e_1 + e_3 + e_4 - e_5 - e_6}{2}$ here to elaborate on the constraints borne by the roots of type $E$ occurring in the projections.

Let us consider a root of type $E$ different from $\alpha_1$ and call it $\beta$, either its scalar product to $\alpha_1$ is -1, either
it is orthogonal to it.

In $E_6$, if $\beta$ is positive since $e_0 - e_7$ is fixed, the product of $e_0 - e_7$ with itself gives 2, and we need the products over all the other indices to sum up to -6. If $\beta$ is negative, the product of $e_0 - e_7$ with $e_7 - e_0$ gives -2, we need the products over all the other indices to sum up to -2 = -4+2, the only option is to have two signs unchanged and four changed within the indices $\{1, \ldots, 6\}$.

The second option is to have $\alpha_1$ orthogonal to $\beta$. If $\beta$ is positive, since the product of $e_0 - e_7$ with itself gives 2, we need the products over all the other indices to sum up to -2, and $-2 = 2 - 4$. If $\beta$ is negative, the product of $e_0 - e_7$ with $e_7 - e_0$ gives -2, and then we need the products over all the other indices to sum up to 2, and $2 = 4 - 2$. Let us consider two examples: the roots $\frac{1}{2}[-e_0 - e_1 + e_2 + e_3 - e_4 - e_5 + e_6 + e_7]$ and $\frac{1}{2}[e_0 + e_1 - e_2 - e_3 + e_4 + e_5 - e_6 - e_7]$ are orthogonal to $\alpha_1$.

In the case of $E_7$ and $E_8$, our sole constraint is that the number of negative signs in the expression of $\beta$ is 4. To obtain $\langle \alpha_1, \beta \rangle = -1 = \frac{-4}{4}$, one needs that among the signs in front of the $e_i$, two signs are the same than in $\alpha_1$ and six change. It is also possible to have $\alpha_1$ orthogonal to $\beta$ when four signs in front of the $e_i$ in the expression of $\beta$ are different from the signs in the expression of $\alpha_1$.

One observes, that at most three roots of the form $\frac{1}{2}[\pm e_0 \pm e_1 \pm e_2 \cdots \pm e_6 \pm e_7]$ can be orthogonal to each other. Therefore if $\Theta$ contains $\alpha_1$, it is still possible to obtain two roots of the form $\frac{1}{2}[\pm e_0 \pm e_1 \pm e_2 \cdots \pm e_6 \pm e_7]$, orthogonal one to another in the projection.

### 3.2.2 Occurrence of type $D$ subsystems in the projection

To observe the occurrence of a type $D_n$ root system in the projection, it is easier to work with the conventions of Bourbaki (see the tables at the end of [1]). We recall here the conventions of Bourbaki for $E_6$ (resp. $E_7$). We consider the hyperplane $\overline{V}$ of $\mathbb{R}^5$ whose points have coordinates $\xi_i$ satisfying $\xi_6 = \xi_7 = -\xi_8$ (resp. orthogonal to $e_7 + e_8$ for $E_7$).

The positive roots are of the following form:

- $\pm e_i + e_j$ for $1 \leq i < j \leq 5$ (resp. $\leq 6$ and along with $(e_8 - e_7)$).
- $\frac{1}{2}[e_8 - e_7 - e_6 \pm e_1 \pm e_2 \cdots \pm e_5]$ with an even number of negative signs. (resp. $\frac{1}{2}[e_8 - e_7 \pm e_1 \pm e_2 \cdots \pm e_5 \pm e_6]$ with an odd number of negative signs).

A system of simple roots is given by:

$$\alpha_1 = \frac{1}{2}[(e_1 + e_8) - (e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7)] \quad \text{and} \quad \alpha_i = [e_{i-1} - e_{i-2}] \quad \text{for} \ 3 \leq i \leq 6$$

(resp $\alpha_i = [e_{i-1} - e_{i-2}]$ for $3 \leq i \leq 7$)

and $\alpha_2 = e_1 + e_2$

In the case of $E_6$ and $\Theta = \{\alpha_6\}$ a root system of type $D_3$ occurs in the projection. This subsystem has a basis made of projections of simple roots in $E_6$. If $\Theta = \{\alpha_i\}$, $i \neq \{1 ; 6\}$, we also obtain $D_3$ subsystem but their basis are not made only of projections of simple roots.

If we consider $E_7$, the case $\Theta = \{\alpha_7\}$ gives a root system of type $D_4$ in the projection whose basis is made of projections of simple roots; the cases $\Theta = \{\alpha_i\}$, $i \neq \{1 ; 7\}$ let also root systems of type $D_4$ appear however their basis are not made only of projections of simple roots.
The roots of $E_8$ are the $\pm e_i \pm e_j$ for $1 \leq i < j \leq 8$ and the $\frac{1}{2}[\pm e_8 \pm e_7 \pm e_1 \pm e_2 \cdots \pm e_5 \pm e_6]$ with an even number of negative signs. If $\Theta$ contains only one simple root (which is not $\alpha_1$) we obtain $D_5$ (when $\Theta = \{\alpha_8\}$, the basis of $D_5$ is made only of projections of simple roots). The root system $D_5$ is not the only option; as one can observe in Section 3.2.5, $D_7$ occurs when $\Theta = \{\alpha_8\}$ while using another basis for $E_8$.

### 3.2.3 The case $E_6$

We come back to the conventions as established in Jean-Pierre Labesse’s unpublished note (see the introduction of this subsection).

We consider the euclidean space $\widetilde{V}$ of dimension 8, equipped with a orthonormal basis indexed by the elements of $\mathbb{Z}/8\mathbb{Z}$

$$\{e_0, e_1, \cdots, e_7\}$$

such that $e_0$ will sometimes be denoted $e_8$. The roots of $E_6$ are the roots in $E_7$ orthogonal to $e_7 - e_8 = -(e_7 + e_0)$ (see the definitions of $E_8$ and $E_7$ in the next subsections). They are of the following form:

- $\pm (e_i - e_j)$ for $1 \leq i < j \leq 6$ or $i = 0$ and $j = 7$.
- $\pm \frac{1}{2}[e_0 - e_7] \pm e_1 \pm e_2 \cdots \pm e_6$

with the same number of + and − sign in the bracket. A system of simple roots is given by

$$\alpha_1 = \frac{1}{2}[e_0 + e_1 + e_2 + e_3 - e_4 - e_5 - e_6 - e_7] \quad \text{and} \quad \alpha_{i+1} = [e_{i+1} - e_i] \quad \text{for} \quad 1 \leq i \leq 5.$$

![Dynkin diagram for E6]

**Remark 3.7.** This depiction is different from the one given in Bourbaki: we have used a subsystem of the system $E_8$ as defined by Bourbaki, except that $\epsilon_8$ is here $e_0$ and that we have an order -and therefore simple roots- which is (are) different(s). In particular, in our convention the roles of $\alpha_1$ and $\alpha_2$ in the Dynkin diagram are inverted. The correspondence is the following:

Our notation $\quad \longleftrightarrow \quad$ Bourbaki’s notation

$$\begin{align*}
\alpha_1 = \frac{1}{2}[e_0 + e_1 + e_2 + e_3 - e_4 - e_5 - e_6 - e_7] & \quad \longleftrightarrow \quad \alpha_2 = \epsilon_1 + \epsilon_2 \\
\alpha_2 = e_2 - e_1 & \quad \longleftrightarrow \quad \alpha_1 = \frac{1}{2}[e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + \epsilon_8] \\
\alpha_{i+1} = e_{i+1} - e_i & \quad \longleftrightarrow \quad \alpha_{i+1} = e_i - e_i - 1 \quad \text{for} \quad 2 \leq i \leq 5
\end{align*}$$

With our writing, it is easily seen that there exists an automorphism $\theta(e_i) = -e_{(7-i)}$; sending $\alpha_{i+1}$ on $\alpha_{7-i}$ for $1 \leq i \leq 5$ and it fixes $\alpha_1$ and $\alpha_4$.

One notices that under this convention, there are no roots of type $D$ (see the beginning of Subsection 3.2 for this terminology) in the root systems of $E_6$ and $E_7$; this is why we dealt with the occurrence of type $D$ root systems in the projection earlier on.
Following 3.3, we know that if $\Theta$ contains only one root, whatever is this one root, the root system we obtained in the projection should always be the same. Below, we illustrate this result, and exhibit basis for the $A_5$ root system appearing in the projection whenever $\Theta$ contains a simple root of $E_6$.

Case $\Theta = \{\alpha_1\}$

The projection of $\Delta - \Theta$ is made of the roots

$$\alpha_i = e_i - e_{i-1} \quad i \text{ in } \{2,3,5,6\} \quad \text{whose squared lengths are 2}$$

and

$$\overline{\alpha_i} = e_i - e_{i-1} + \frac{e_0 + e_1 + e_2 + e_3 - e_4 - e_5 - e_6 - e_7}{4} \quad \text{whose squared length is } 3/2.$$

A root system of type $A_5$ appears in the projection, a basis is given by:

$$\{\alpha_4, \alpha_5, \frac{e_0 - e_7 + e_1 - e_2 - e_3 + e_4 + e_5 - e_6}{2}, \alpha_2, \alpha_3\}.$$

Case $\Theta = \{\alpha_2\}$

These form a root system of type $A_4$.

Could we complete this root system to obtain a root system of rank 5?

We are looking for a root $\beta$ which is orthogonal to any $\alpha_i$ with $i \in \{1, 2, 4, 5\}$, whose scalar product with $\alpha_6$ is $-1$ then $\beta = \frac{1}{2}[e_0 - e_1 - e_2 + e_3 + e_4 + e_5 - e_6 - e_7]$ satisfies these conditions; further the sum (which is the longest root) $\beta + \alpha_1 + \alpha_4 + \alpha_5 + \alpha_6 = e_0 - e_7$ is obtained in the projection. Therefore a root system of type $A_5$ is obtained.

Looking at this case in $E_7$, one also observes that another basis of an $A_5$ root system in the projection can be obtained from the roots $\{\alpha_1, \alpha_4, \alpha_5, \alpha_6, \alpha_7 = e_7 - e_6\}$.

The symmetrical case with $\Theta = \{\alpha_6\}$ yields a root system of type $A_5$ whose basis is constituted of $\overline{\alpha_i} = \alpha_i, \ i \in \{1,4\}$ and $\beta = \frac{1}{2}[e_0 + e_1 - e_2 - e_3 - e_4 + e_5 + e_6 - e_7]$. The sum of the simple roots yields $e_0 - e_7$, which appears in the projection.

Case $\Theta = \{\alpha_3\}$

$$\overline{\alpha_i} = \alpha_i, \ i \in \{1,5,6\} \quad \text{whose squared length is 2}.$$
Projection of root systems

Since $e_3 = e_2$, we have:

$$\overline{\alpha_2} = \frac{e_2 + e_3}{2} - e_1 \quad \text{and} \quad \overline{\alpha_4} = e_4 - \frac{e_2 + e_3}{2}$$
whose squared lengths are $3/2$.

The roots $\overline{\alpha_5}$ and $\overline{\alpha_3}$ cannot form a root system since $C = 3$. The roots \{\(\alpha_1 = \frac{1}{2}[e_0 + e_1 + e_2 + e_3 - e_4 - e_5 - e_6 - e_7]; e_4 - e_1, \alpha_5, \alpha_6, \alpha_7\)\} constitute the basis of an $A_5$.

The symmetrical case $\Theta = \{\alpha_3\}$ is treated similarly.

Case $\Theta = \{\alpha_4\}$

Since $e_4 = e_3$, we have:

$$\overline{\alpha_1} = \frac{[e_0 + e_1 + e_2 - e_5 - e_6 - e_7]}{2} \quad \text{whose squared length is } 3/2,$$

$$\overline{\alpha_3} = \frac{[e_3 + e_4]}{2} - e_2 \quad \text{and} \quad \overline{\alpha_5} = e_5 - \frac{1}{2}[e_3 + e_4]$$
whose squared lengths are $3/2$.

Considering the roots $\overline{\alpha_3}$ and $\overline{\alpha_5}$ (or $\overline{\alpha_3}$ and $\overline{\alpha_1}$), the value of $C = 9$, whereas for $\overline{\alpha_7}$ and $\overline{\alpha_2}$ (resp. $\overline{\alpha_5}$ and $\overline{\alpha_6}$) it is 3.

A root system of type $A_5$ appears in the projection, its basis is given by

$$\{\alpha_6, e_5 - e_2, e_2 - e_1, \frac{e_0 - e_7 + e_1 - e_2 + e_3 + e_4 - e_5 - e_6}{2}, e_7 - e_0\}$$

Case $\Theta = \{\alpha_1, \alpha_4\}$

Since $e_4 = e_3$, we have:

$$\overline{\alpha_1} = \frac{[e_3 + e_4]}{2} - e_2 + \frac{[e_0 + e_1 + e_2 - e_5 - e_6 - e_7]}{6}$$

$$\overline{\alpha_5} = e_5 - \frac{[e_3 + e_4]}{2} + \frac{[e_0 + e_1 + e_2 - e_5 - e_6 - e_7]}{6}.$$

The squared length of $\overline{\alpha_3}$ (resp. $\overline{\alpha_5}$) is $51/32$ and the value of $C$, considered with respect to $\overline{\alpha_2}$ (resp. $\overline{\alpha_6}$), does not correspond to any root system. Although we could compose a root system from the elements \{\(e_0, e_1, e_2, e_5, e_6, e_7\)\}, as opposed to the context of $E_7$, one cannot add $e_7 - e_0$ to $\alpha_6$ since,
expressions of the form $e_0 - e_i$, $i \neq 7$ (for instance the sum $e_7 - e_0$ and $\alpha_6$ which equals $e_0 - e_6$) are not roots of $E_6$. However, we can use root of the form $\frac{1}{2}[e_0 \pm e_1 \pm e_2 \cdots \pm e_6 - e_7]$ orthogonal to $\alpha_1$. Then, a root system of type $A_3$ with basis $\{\alpha_6, \frac{1}{2}[e_0 + e_1 - e_2 + e_3 + e_4 + e_5 - e_6 - e_7], e_2 - e_1\}$ appears in the projection.

We now consider the cases $\Theta = \{\alpha_1, \alpha_5\}$ and $\Theta = \{\alpha_1, \alpha_6\}$. The corresponding symmetrical cases, $\Theta = \{\alpha_1, \alpha_3\}$ and $\Theta = \{\alpha_1, \alpha_2\}$, imply clearly the same reasoning and results.

Case $\Theta = \{\alpha_1, \alpha_5\}$ (resp. $\{\alpha_1, \alpha_3\}$)

$$\alpha_1 = \frac{e_4 + e_5}{2} - e_3 + e_0 + e_1 + e_2 + e_3 - e_4 - e_5 - e_6 - e_7$$

The value of $C$ when considering $\overline{\alpha_1}$ and $\overline{\alpha_3}$ is 4.

The roots $\overline{\alpha_2}, \overline{\alpha_3}, \overline{\alpha_4}$ constitute a basis for a root system of type $A_3$. However, the longest root $\overline{\alpha_1} + \overline{\alpha_3} + \overline{\alpha_2} = \frac{1}{4}[e_0 + e_1 + e_2 + e_3 + e_4 + e_5 - e_6 - e_7]$ is not a projection of a root in $\Sigma$. Therefore, a subsystem in $\Sigma_{69}$ is $A_2$ with basis $\{\overline{\alpha_2}, \overline{\alpha_3}\}$. It is possible to find a root $\beta$ (of the form $\pm \frac{1}{2}(e_0 - e_7) \pm e_1 \pm e_2 \cdots \pm e_6$) which is orthogonal to $\alpha_i$ for $i \in \{1, 3, 5\}$ and such that its scalar product with $\alpha_2$ is -1: $\frac{1}{2}[e_0 + e_1 - e_2 - e_3 + e_4 + e_5 - e_6 - e_7]$. The sum of this root with $\alpha_2$ and $\alpha_3$ appears in the projection. Therefore the subsystem of highest rank in $\Sigma_{69}$ is $A_3$.

Case $\Theta = \{\alpha_1, \alpha_6\}$ (resp. $\{\alpha_1, \alpha_2\}$)

$$\overline{\alpha_i} = \alpha_i \quad i \in \{2, 3\}$$

Since $\overline{\alpha_6} = \overline{\alpha_5}$, we have:

$$\overline{\alpha_5} = \frac{e_0 + e_5}{2} - e_4$$

$$\overline{\alpha_4} = \frac{e_4 - e_3 + \frac{1}{4}[e_0 + e_1 + e_2 + e_3 - e_4 - e_5 - e_6 - e_7].$$

The squared length of $\overline{\alpha_4}$ is $2 + 1/2 - 4.1/4 = 3/2$; therefore when considering $\overline{\alpha_4}$ and $\overline{\alpha_3}$, $C = 3$. This value of $C$ does not correspond to any root system of rank 2, therefore we need to exclude the possibility of rank 3 system (when completing those two roots with $\overline{\alpha_3}$). A subsystem $A_2$ has basis given by $\overline{\alpha_3}$ and $\overline{\alpha_2}$.

It is possible to find a root $\beta$ which is orthogonal to $\alpha_i$ for $i \in \{1, 3, 6\}$ and such that its scalar product with $\alpha_2$ is -1: $\frac{1}{2}[e_0 + e_1 - e_2 - e_3 - e_4 + e_5 + e_6 - e_7]$; its sum with $\alpha_2$ and $\alpha_3$ appears in the projection. Therefore the subsystem of highest rank in $\Sigma_{69}$ is $A_3$.

Case $\Theta = \{\alpha_2, \alpha_6\}$

$$\overline{\alpha_i} = \alpha_i \quad i \in \{1, 4\}$$

whose squared length is 2,
\[ \overline{\alpha_3} = e_3 - \frac{1}{2}[e_1 + e_2], \]
\[ \overline{\alpha_5} = \frac{1}{2}[e_6 + e_5] - e_4. \]

Consider \( \overline{\sigma_1} \) and \( \overline{\sigma_4} \), their scalar product is -1. \( R = 1, C = 4 \). This could give us \( A_2 \).

The root \( \overline{\sigma_1} + \overline{\sigma_4} = \frac{1}{2}[e_0 + e_1 + e_2 - e_3 + e_4 - e_5 - e_6 - e_7] \) appears in the projection. Considering the value of \( C \) between (for instance) \( \overline{\sigma_1} \) and \( \overline{\sigma_4} \) yields 3 which forbids the appearance of a root system of higher rank. However, the root \( \frac{1}{2}[e_0 - e_1 - e_2 + e_3 - e_4 + e_5 + e_6 - e_7] \) orthogonal to \( \alpha_i \) for \( i \in \{1, 2, 6\} \) is the third root which constitute with \( \overline{\sigma_1} \) and \( \overline{\sigma_4} \) the basis of a root system of type \( A_3 \). This subsystem is of highest rank.

Case \( \Theta = \{\alpha_2, \alpha_5\} \)

\[ \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet
\end{array} \]

Using (in particular) Lemma 3.6, we see that the ratio of lengths of \( \overline{\alpha_3} \) and \( \overline{\alpha_4} \) is 1 ; whereas their scalar product is -1, and \( C = 9/4 \). Considering now \( \overline{\sigma_1} \) and \( \overline{\sigma_4} \), one has a scalar product of -1, a ratio \( R \) of 4/3 and \( C = 3 \). In both cases, the value of \( C \) does not correspond to any rank 2 root system. Further, in the projection, we also obtain \( \beta = \frac{1}{2}[e_0 - e_3 + e_6 - e_7] \) of squared norm 1 ; and \( \beta' = \frac{1}{2}[e_0 - e_1 + e_2 - e_3 + e_6 - e_7] \) or \( \beta'' = \frac{1}{2}[e_0 - e_3 + e_4 + e_5 + e_6 - e_7] \) of squared norm 3/2. When looking at scalar product of, say \( \beta' \) and \( \alpha_3 \), we also reach a value of \( C = 9/4 \). However, it is possible to find a system of type \( A_3 \), using \( \{e_0, e_7, e_3, e_6\} \) and root of the form \( \pm \frac{1}{2}[e_0 - e_7 \pm e_1 \pm e_2 \cdots \pm e_6] \). For instance, a basis is given by : \( \{e_7 - e_0, \frac{1}{2}[e_0 - e_7 + e_1 + e_2 - e_5 - e_4 + e_6 - e_3], e_3 - e_6\} \).

Case \( \Theta = \{\alpha_3, \alpha_5\} \)

\[ \begin{array}{c}
\bullet \\
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\bullet \\
\bullet \\
\bullet
\end{array} \]

\[ \overline{\alpha_4} = \frac{[e_5 + e_4]}{2} - \frac{[e_3 + e_2]}{2}, \]
\[ \overline{\alpha_2} = \frac{[e_3 + e_2]}{2} - e_1, \]
\[ \overline{\alpha_6} = e_6 - \frac{[e_5 + e_4]}{2}. \]

The squared norm of \( \overline{\sigma_1} \) is 2, whereas the squared norm of \( \overline{\sigma_4} \) is 1. The squared norms of \( \overline{\sigma_2}, \overline{\sigma_6} \) is 3/2. The scalar product \( \langle \overline{\sigma_1}, \overline{\sigma_4} \rangle = -1, R = 2 \) and \( C = 2 \), these roots form the basis of a root system of type \( B_2 \).

Notice that \( \overline{\sigma_1} + 2\overline{\sigma_4} = \frac{e_0 + e_1 - e_2 - e_3 + e_4 + e_5 - e_6 - e_7}{2} \) (remaining the same when projected since orthogonal to \( \alpha_3, \alpha_5 \)) is in \( \Sigma_0 \) and \( \overline{\sigma_1} + \overline{\sigma_4} \), projection of \( \frac{e_0 + e_1 - e_2 - e_3 + e_4 + e_5 - e_6 - e_7}{2} \) also. Therefore a root system of type \( B_2 \) appears in the projection. There also is a \( A_3 \) root system whose basis is, for instance, given by : \( \{e_7 - e_0, \frac{1}{2}[e_0 - e_7 + e_1 + e_2 - e_5 + e_4 - e_6 + e_1], e_6 - e_1\} \).

Case \( \Theta = \{\alpha_2, \alpha_3\} \cup \{\alpha_5, \alpha_6\} \)

\[ \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet
\end{array} \]
**Lemma 3.8.** For a system of type $E_6$, if $\Theta$ is the union of two components of type $A_2$ defined by $\{\alpha_2, \alpha_3\}$ and $\{\alpha_5, \alpha_6\}$, the projection on the orthogonal of $\Theta$ contains the basis of a system of $G_2$ but not the whole system.

**Proof.** The projections are:

$$\overline{\alpha_1} = \alpha_1 \quad \text{and} \quad \overline{\alpha_4} = \frac{\alpha_4 + \alpha_5 + \alpha_6}{3} - \frac{\alpha_1 + \alpha_2 + \alpha_3}{3},$$

where $||\overline{\alpha_1}||^2 = 2$ and $||\overline{\alpha_4}||^2 = 6/9 = 2/3$ and the scalar product is $-1$. Therefore $R = 3$ and $C = 4/3$ and this is the basis of a root system of type $G_2$. For the projection to contain a system of type $G_2$ one would need that we obtain $\overline{\alpha_1} + \overline{\alpha_4}$, $\overline{\alpha_1} + 2\overline{\alpha_4}$, $\overline{\alpha_1} + 3\overline{\alpha_4}$ and $2\overline{\alpha_1} + 3\overline{\alpha_4}$. But

$$\overline{\alpha_1} = \frac{1}{2}[e_0 + e_1 + e_2 + e_3 - e_4 - e_5 - e_6 - e_7] = \frac{1}{2}[(e_0 - e_7) - 3e_2]$$

and we check easily that $\overline{\alpha_1} + 3\overline{\alpha_4}$ is obtained by varying the signs in the parenthesis. The root $2\overline{\alpha_1} + 3\overline{\alpha_4} = e_0 - e_7$ is also obtained. However, $\overline{\alpha_1} + \overline{\alpha_4}$ (for instance) is not obtained and therefore $G_2$ is not a subsystem.

Case $\Theta = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5\}$

The value of $C$ corresponding to those two roots is $25$. Therefore no root system can be obtained in the projection.
Table 2: Roots system occurring in Σθ for Σ of type $E_6$

Root systems of type $D$ occurring, in particular when $\Theta$ contains only one root, are not systematically written.

### 3.2.4 The case $E_7$

We consider as for $E_6$, an euclidean space $\tilde{V}$ of dimension 8, equipped with an orthonormal basis indexed by the elements of $\mathbb{Z}/8\mathbb{Z}$

$$\{e_0, e_1, \cdots, e_6, e_7\}$$

The roots of $E_7$ are the roots of $E_8$ orthogonal to $e_7 + e_8 = (-1/2)(e_0 + e_2 + \cdots + e_7)$ where $e_7, e_8$ refer to the notations in [1] and the change of basis was given in an unpublished note of Jean-Pierre Labesse. They are of the following form:

- $\pm(e_i - e_j)$ for $0 \leq i < j \leq 7$.
- $\frac{1}{2}[\pm e_0 \pm e_7 \pm e_1 \pm e_2 \cdots \pm e_6]$
Projection of root systems

with a number of + signs (and therefore of −) in the bracket equal to 4. A root system is given by:

\[ \alpha_1 = \frac{1}{2}[e_0 + e_1 + e_2 + e_3 - e_4 - e_5 - e_6 - e_7] \quad \text{and} \quad \alpha_{i+1} = [e_{i+1} - e_i] \quad \text{pour} \ 1 \leq i \leq 6. \]

Following 3.3, we know that if Θ contains only one root, whatever is this one root, the root system we obtained in the projection should always be the same. Below, we illustrate this result, and exhibit basis for the \( A_5 \) root system appearing in the projection whenever Θ contains a simple root of \( E_7 \).

Case \( \Theta = \{\alpha_1\} \)

\[ \alpha_i = \alpha_i \quad \text{for} \ i \in \{2, 3, 5, 6, 7\}, \]
\[ \alpha_4 = e_4 - e_3 + \frac{e_0 + e_1 + e_2 + e_3 - e_4 - e_5 - e_6 - e_7}{4}. \]

The squared norm of \( \alpha_i \) is 3/2, whereas the squared norm of \( \alpha_i \) for \( i \) in \( \{2, 3, 5, 6, 7\} \) is 2. The system of greatest rank in the projection while restricting only on projection of simple roots is one which is of type \( A_3 \) with basis \( \alpha_i \) for \( i \) in \( \{5, 6, 7\} \). The roots \( \{\alpha_6, \alpha_7, e_0 + e_1 - e_2 - e_3 - e_4 + e_5 + e_6 - e_7, e_2 - e_1, e_1 - e_0\} \) constitute the basis of a root system of type \( A_5 \), the sums of roots appear in the projection, hence we obtain a root system of type \( A_5 \).

Case \( \Theta = \{\alpha_2\} \)

The projection of \( \Delta - \Theta \) is made of the \( \alpha_i = e_i - e_{i-1} \), \( i \) in \( \{1, 4, 5, 6, 7\} \) which constitute the basis of a root system of type \( A_5 \), and \( \alpha_3 = e_3 - \frac{e_1 + e_2}{2} \).

There does not exist any root \( \beta \), of the form \( \frac{1}{2}[\pm e_0 \pm e_7 \pm e_1 \pm e_2 \cdots \pm e_6] \), which is orthogonal to \( \alpha_i \) for \( i \) in \( \{1, 2, 4, 5, 6\} \) and whose scalar product with \( \alpha_i \) is -1, therefore we cannot complete this system to form a system of rank 6 ; hence it is of highest rank.

We could also imagine that a root \( \beta \) (of the form \( \pm \frac{1}{2}[\pm e_0 \pm e_7 \pm e_1 \pm e_2 \cdots \pm e_6] \)) to complete the basis—would be orthogonal to all \( \alpha_i \), \( i \) in \( \{2, 4, 5, 6, 7\} \) and its scalar product with \( \alpha_1 = \alpha_1 \) be -1. Such root does not exist. The scalar product is necessarily one.

Case \( \Theta = \{\alpha_3\} \)

\[ \alpha_i = \alpha_i \quad i \in \{1, 5, 6, 7\} \] whose squared length is 2.

Since \( \alpha_3 = \alpha_2 \), we have : \( \alpha_2 = \frac{e_0 + e_1}{2} - e_1 \) and \( \alpha_4 = e_4 - \frac{e_3 + e_1}{2} \) whose squared length are 3/2.
Roots of the form \(\pm(e_i - e_j)\) can be made out of the elements \(\{e_0, e_1, e_4, e_5, e_6, e_7\}\); they form a root system of type \(A_5\). It cannot be completed by any root of the form \(\frac{1}{2}[\pm e_0 \pm e_7 \pm e_1 \pm e_2 \cdots \pm e_6]\) without getting a contradiction to the requirement of having 4 positive and negative signs within the bracket.

Case \(\Theta = \{\alpha_4\}\)

Since \(\alpha_4 = e_4\), we have:

\[
\frac{\alpha_1}{1} = \frac{e_0 + e_1 + e_2 - e_5 - e_6 - e_7}{2}
\]

whose squared length is \(3/2\).

Moreover,

\[
\alpha_3 = \frac{e_3 + e_4}{2} - e_2,
\]

and

\[
\alpha_5 = e_5 - \frac{e_3 + e_4}{2}
\]

whose squared length is \(3/2\).

\[
\alpha_i = \alpha_i \quad i \text{ in } \{2, 6, 7\}.
\]

Same reasoning than in the previous case, we obtain an \(A_5\) root system in the projection.

The case \(\Theta = \{\alpha_5\}\) and \(\Theta = \{\alpha_7\}\) are treated similarly and yield the same result. The ratios of lengths do not allow the occurrence of root systems of type \(G_2\), \(F_4\) and \(E_6\). The roots \(\{e_7 - e_6, e_0 - e_7, e_1 - e_0, e_2 - e_1, e_3 - e_2\}\) (resp. \(\{e_1 - e_0, e_2 - e_1, e_3 - e_2, e_4 - e_3, e_5 - e_4\}\)) constitute the basis of a root system of type \(A_5\). Since all the sum of any consecutive roots in this basis appear in the projection, we obtain a root system of type \(A_5\) in the projection.

Case \(\Theta = \{\alpha_5, \alpha_6, \alpha_7\}\)

\[
\alpha_i = \alpha_i \quad i \text{ in } \{1, 2, 3\}, \text{ whose squared length is } 2 ; \alpha_4 = \frac{e_4 + e_5 + e_6 + e_7}{4} - e_3 \text{ whose squared length is } 5/4.
\]

The ratios of lengths are incompatible with \(F_4\). The roots \(\{e_1 - e_0, e_2 - e_1, e_3 - e_2\}\) constitute the basis of the \(A_3\) root system occurring in the projection.

Case \(\Theta = \{\alpha_2, \alpha_3\} \cup \{\alpha_5, \alpha_6, \alpha_7\}\)

\[
\alpha_i = \alpha_i \text{ with squared length } 2 ; \alpha_4 = \frac{e_4 + e_5 + e_6 + e_7}{4} - e_3 \text{ with squared length } 7/12.
\]

The ratios of lengths are incompatible with a root system of type \(G_2\), and even any classical root system of rank 2.

Case \(\Theta = \{\alpha_3, \alpha_5\}\)

\[
\alpha_i = \alpha_i \quad i \text{ in } \{2, 6, 7\}, \text{ whose squared length is } 1.
\]
The value of $C$ when considering the projected roots $\overline{\alpha_4}$ and $\overline{\alpha_2}$ is 8. The roots $\overline{\alpha_4}$ and $\overline{\alpha_7}$ constitute the basis of a root system of type $B_2$. Since $\overline{\alpha_4} + \overline{\alpha_1}$ and $2\overline{\alpha_4} + \overline{\alpha_1}$ are obtained in the projection, a system of type $B_2$ appear. We can complete the basis with $e_7 - e_0$ and $e_0 - e_1$ to obtain a root system of type $B_4$. Notice also that the roots $\{e_6 - e_7, e_7 - e_1, e_1 - e_0\}$ constitute the basis of the $A_3$ root system occurring in the projection. To add a root of the form $\frac{1}{2}[\pm e_0 \pm e_7 \pm e_1 \pm e_2 \cdots \pm e_6]$ to this basis, it would need to have the same sign for all $e_i, i \in \{2, 3, 4, 5\}$ and one other $e_i, i \in \{0,7\}$ or $\{7,1\}$, this contradicts the requirement of having 4 positive and negative signs in the bracket.

Case $\Theta = \{\alpha_2\} \cup \{\alpha_6, \alpha_7\}$

$$\overline{\alpha_5} = \frac{e_5 + e_6 + e_7}{3} - e_4 \quad \text{whose squared length is } 4/3.$$  

$$\overline{\alpha_3} = e_3 - \frac{e_2 + e_1}{2} \quad \text{whose squared length is } 3/2.$$  

$$\overline{\alpha_1} = \alpha_i, \text{ for } i=1 \text{ or } 4, \text{ whose squared length is } 2.$$  

The ratios of lengths are incompatible with $F_4$ or even any classical system of rank 4. Using Lemma 3.6, it is clear that no root system can be obtained from $\overline{\alpha_3, \alpha_4, \alpha_7}$. Therefore the only root system one can obtain is $A_2$ with basis $\overline{\alpha_i} = \alpha_i$ for $i = 1$ and 4.

Case $\Theta = \{\alpha_1, \alpha_7\}$

The roots $\{e_1 - e_0, e_2 - e_1, \frac{1}{2}[e_0 + e_1 - e_2 - e_3 + e_4 + e_5 - e_6 - e_7], e_3 - e_2\}$ constitute the basis of the $D_4$ root system occurring in the projection.

For the cases $\Theta = \{\alpha_1, \alpha_i\}$, with $i \in \{4,5\}$, we obtain $A_3$ root systems. A basis is given by $\{\alpha_2, \alpha_3, \frac{1}{2}[-e_0 + e_1 + e_2 - e_3 - e_4 - e_5 + e_6 + e_7]\}$.

The case $\Theta = \{\alpha_1, \alpha_2\}$ (resp. $\Theta = \{\alpha_1, \alpha_3\}$) gives rise to a root systems of type $A_4$ (resp. $A_3$) in the projection (the argumentation is similar to the one for $E_6$). It is not possible to find a root whose scalar product with $\alpha_7$ is -1 and which is orthogonal to $\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6$ (resp to $\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6$) to complete this $A_4$ (resp. $A_3$) to an $A_5$ (resp. $A_4$). Therefore, the root system $A_4$ (resp. $A_3$) is of highest rank in the projection.

Case $\Theta = \{\alpha_2, \alpha_7\}$

Treated similarly than the case $E_6, \Theta = \{\alpha_2, \alpha_6\}$. We use Lemma 3.6 with $\{\overline{\alpha_3, \alpha_4}\}$ and $\{\overline{\alpha_6, \alpha_7}\}$. The roots $\{\overline{\alpha_5, \alpha_7, \alpha_1}\}$ constitute the basis of a root system of type $A_3$. It is not possible to find a root of the form $\frac{1}{2}[\pm e_0 \pm e_7 \pm e_1 \pm e_2 \cdots \pm e_6]$ which is orthogonal to any $\alpha_i$ for $i \in \{1, 2, 4, 7\}$ and whose scalar product with $\overline{\alpha_5} = \alpha_5$ is -1; The root system of type $A_3$ is of highest rank in the projection.
Case $\Theta = \{\alpha_2, \alpha_4, \alpha_6\}$
We have $\alpha_3 = \frac{\alpha_2 + \alpha_4}{2} - \frac{\alpha_2 - \alpha_4}{2}$ and $\alpha_5 = \frac{\alpha_2 + \alpha_6}{2} - \frac{\alpha_2 - \alpha_6}{2}$ of squared norms equal to one. Further $C = 4$, and $R = 1$ hence this give us the basis of an $A_2$. We can complete this with $\frac{1}{2}[-e_0 - e_7 + e_1 + e_2 + e_3 + e_4 - e_5 - e_6]$, which has scalar product -1 with $\alpha_3$ and $\frac{1}{2}[+e_0 + e_7 + e_1 + e_2 - e_3 - e_4 - e_5 - e_6]$, which has scalar product -1 with $\alpha_5$. The sum of these basis roots is $\frac{\alpha_2 + \alpha_6}{2} - \frac{\alpha_2 - \alpha_6}{2}$ and therefore appears in the projection. The case $\Theta = \{\alpha_2, \alpha_4, \alpha_6\}$ only gives an $A_2$.

Case $\Theta = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5\}$ is treated similarly than in $E_6$. The squared norm of $\alpha_2$ and $\alpha_6$ is $5/4$; while $C = 25$; hence roots $\alpha_2$ and $\alpha_6$ do not form a root system. Also, notice that among the roots formed from the vectors $\{e_1, e_0, e_7\}$ only $e_6 - e_7$ is orthogonal to $\alpha_1$.

Case $\Theta = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$

Although the vectors $\{e_1, e_0, e_7\}$ could constitute some roots of type $A$, only one of them, $e_1 - e_0$, is orthogonal to $\alpha_1$.

The cases of $\Theta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and $\Theta = \{\alpha_2, \alpha_3, \alpha_4\}$ are treated simultaneously. We have a root system of type $A_2$ and basis $\alpha_6, \alpha_7$. We cannot add a root of the form $\frac{1}{2}[\pm e_0 \pm e_7 \pm e_1 \pm e_2 \cdots \pm e_6]$ since it would have to get the same sign for all $e_i, i \in \{1, 2, 3, 4\}$ and one $e_i, i \in \{6, 5\}$ or $\{6, 7\}$, this contradicts the requirement of having 4 positive and negative signs in the bracket.
<table>
<thead>
<tr>
<th>$\Theta = {.}$</th>
<th>squared lengths of projected roots</th>
<th>chosen roots to calculate $C$ and $R$</th>
<th>$C$ and $R$</th>
<th>root system of highest rank obtained (of rank $\geq 2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_1$</td>
<td>2 and 5/2</td>
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<td></td>
<td>$A_5$</td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>2 and 3/2</td>
<td></td>
<td></td>
<td>$A_5$</td>
</tr>
<tr>
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<td>2 and 3/2</td>
<td>$\alpha_5, \alpha_1$</td>
<td>$C=3/2$</td>
<td>$A_5$</td>
</tr>
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<td>$\alpha_4$</td>
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<td>$(\alpha_1, \alpha_3) : (\alpha_2, \alpha_3)$</td>
<td>$C=9 ; C=3$</td>
<td>$A_5$</td>
</tr>
<tr>
<td>$\alpha_5$</td>
<td>2 and 3/2</td>
<td>$\alpha_5, \alpha_1$</td>
<td>$C=3$</td>
<td>$A_5$</td>
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<tr>
<td>$\alpha_6$</td>
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<tr>
<td>$\alpha_7$</td>
<td></td>
<td></td>
<td></td>
<td>$A_5$ or $D_4$</td>
</tr>
<tr>
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<td>4/3, 3/2 and 2</td>
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</tr>
<tr>
<td>$\alpha_5, \alpha_6, \alpha_7$</td>
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<tr>
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<tr>
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<td>$\alpha_1, \alpha_4$</td>
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<td></td>
<td></td>
<td>$A_2$</td>
</tr>
<tr>
<td>$\alpha_2, \alpha_4, \alpha_6$</td>
<td></td>
<td></td>
<td></td>
<td>$A_4$</td>
</tr>
</tbody>
</table>

Table 3: Roots system occurring in $\Sigma_{e_8}$ for $\Sigma$ of type $E_7$

Root systems of type $D$ occurring, in particular when $\Theta$ contains only one root, are not systematically written.
3.2.5 The case $E_8$

The positive roots are of the following form:

- $\pm e_i \pm e_j$ for $0 \leq i < j \leq 7$.
- $\frac{1}{2}[\pm e_0 \pm e_1 \pm e_2 \cdots \pm e_6 \pm e_7]$ with an even number of negative signs.

A system of simple roots is given by:

$$\alpha_1 = \frac{1}{2}[e_0 + e_1 + e_2 + e_3 - e_4 - e_5 - e_6 - e_7] \quad \text{and} \quad \alpha_i = e_i - e_{i-1} \quad \text{for } 2 \leq i \leq 8$$

\begin{center}
\[ \begin{array}{c}
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \alpha_8 \\
\end{array} \]
\end{center}

3.2.6 Cases with $\Theta$ containing only one element

Let us assume $\Theta$ contains $\alpha_k = e_k - e_{k-1}$ (resp. $\alpha_1$ or $\alpha_8$). Let $i,j \neq k,k-1$ (resp. $i,j \neq 3,4$ or $i,j \neq 7,6$). In the projection, the roots of the form $\pm e_i \pm e_j$ have squared norms equal to 2. The projections of roots of the form $\pm e_k \pm e_{k+1}$ and $\frac{1}{2}[\pm e_0 \ldots \pm e_{k-2} \ldots \pm e_{k+1} \ldots \pm e_7]$ have squared norms equal to $3/2$. The ratio of lengths do not allow $F_4$ and $G_2$ since the squared norms of projected roots are 2 or $3/2$. Since there are no roots of norms 1, or 4, by the remark in Subsection 1.2, the ratio of lengths allow only the occurrence of root systems of type $A$ and $D$.

Following 3.3, we know the nature of the root system of maximal rank occurring in the projection does not depend on the choice of root in $\Theta$. This explains the statement of Theorem 1.2. Indeed,

Case $\Theta = \{\alpha_8\}$

\begin{center}
\[ \begin{array}{c}
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \alpha_8 \\
\end{array} \]
\end{center}

We obtain $\alpha_1 = \frac{1}{2}[e_0 + e_1 + e_2 + e_3 - e_4 - e_5 - e_6 - e_7]$ and the $\alpha_i$ with $2 \leq i \leq 6$ which generates the $E_6$, but also $\beta = -\frac{1}{2}[e_0 - e_1 - e_2 - e_3 - e_4 - e_5 + e_6 - e_7]$. The Dynkin diagram associated to $(\alpha_1, \ldots, \alpha_6, \beta)$ is the one of $E_7$.

Remark 3.9. This phenomenon is specific to $E_8$; Recall that with the conventions of [1] the roots of $E_7$ are the roots of $E_8$ orthogonal to the root $e_7 + e_8$ of $E_8$. The roots of $E_6$ are the orthogonal in $E_7$ to $\pi = e_6 + e_7 + 2e_8$ which is not a root in $E_7$. Hence the phenomenon observed in the previous point does not occur: we cannot obtain $E_6$ when projecting orthogonally to a unique (simple) root in $E_7$.

We also give the basis of an $A_7$, and $D_6$ in the case $\Theta = \{\alpha_2\}$. The roots $\alpha_i = \alpha_i$ for $i \in \{1, 4, 5, 6, 7, 8\}$ together with $\frac{1}{2}[e_0 + e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7]$ form the basis of a root system of type $A_7$. One can check that the sum of these roots is $e_1 + e_2$ which appears in
the projection.
Let us consider the occurrence of type $D$ root system in this case. With this choice of basis, we only get a $D_6$ in the projection. The roots $\pm e_i \pm e_j$ for $i < j$ in $\{0, 3, 4, 5, 6, 7\}$ constitute the roots of $D_6$. The root $\frac{1}{2}[e_6 + e_1 + e_2 + e_3 - e_4 - e_5 - e_6 - e_7]$ is orthogonal to all $\alpha_i$ for $i \in \{\alpha_4, \ldots, \alpha_7\}$ but it cannot be orthogonal to $e_0 + e_7$ and $e_7 - e_0$ which could constitute the two other extremal roots of the Dynkin diagram. Adding roots of the form $\frac{1}{2}[\pm e_0 \pm e_1 \pm e_2 \cdots \pm e_6 \pm e_7]$, orthogonal between themselves and with all others but one in the basis, as extremal roots of the Dynkin diagram also leads to contradiction: The sum of all the basis roots does not appear in the projection.

3.2.7

Case $\Theta = \{\alpha_1, \alpha_2\}$

The inverse of the projections $\overline{\alpha_i} = \alpha_i$, $i \in \{5, 6, 7, 8\}$ form the basis of a root system of type $A_4$. We are looking for a root $\beta$ of the form $\frac{1}{2}[\pm e_0 \pm e_1 \pm e_2 \cdots \pm e_6 \pm e_7]$ (with an even number of negative signs) which is orthogonal to all $\alpha_i$, $i \in \{1, 2, 5, 6, 7\}$ and whose scalar product with $\overline{\alpha_i} = \alpha_8$ is -1. The root $\beta = \frac{1}{2}[e_0 + e_7 + e_1 + e_2 + e_3 + e_4 + e_5 + e_6]$ satisfies this condition. This root completes the basis for a root system of type $A_5$. The sum of the simple roots is $\frac{1}{2}[e_0 + e_7 - e_1 - e_2 + e_3 + e_4 + e_6]$, which appears in the projection. One could also complete the above basis for $A_4$ with the root $e_0 - e_3$ to obtain a root system of type $A_5$.

Case $\Theta = \{\alpha_1, \alpha_3\}$

The roots $\overline{\alpha_i} = \alpha_i$ for $i \in \{5, \cdots, 8\}$ along with $e_0 - e_1$ and $e_1 - e_2$ form the basis of a root system of type $A_6$.

Case $\Theta = \{\alpha_1, \alpha_4\}$

The roots $\overline{\alpha_i} = \alpha_i$ for $i \in \{6, 7, 8\}$ along with $e_0 - e_1$, $e_1 - e_2$, and $e_2 - e_3$ form the basis of a root system of type $A_6$.

Case $\Theta = \{\alpha_1, \alpha_5\}$

The roots $\overline{\alpha_i} = \alpha_i$ for $i \in \{7, 8\}$ along with $e_0 - e_1$, $e_1 - e_2$ and $e_2 - e_3$ form the basis of a root system of type $A_5$.

Case $\Theta = \{\alpha_1, \alpha_6\}$
The roots $\alpha_i$ for $i \in \{2, 3\}$ along with $e_1 - e_0$, $e_0 + e_7$ form the basis of a root system of type $A_4$. It is not possible to find a root of the form $\frac{1}{2}[\pm e_0 \pm e_1 \pm e_2 \cdots \pm e_6 \pm e_7]$ (with an even number of negative signs) which is orthogonal to all $\alpha_i$, $i \in \{1, 2, 3, 6\}$, and $e_1 - e_0$ (resp. $\alpha_i$, $i \in \{1, 2, 6\}$, $e_1 - e_0$, and $e_0 + e_7$) and whose scalar product with $e_7 + e_0$ (resp $e_3 - e_2$) is -1.

Case $\Theta = \{\alpha_1, \alpha_7\}$

The roots $e_i - e_{i+1}$ for $i \in \{0, 1, 2\}$, $e_3 + e_4$ and $e_5 - e_4$ form the basis of a $A_5$ root system.

For $\Theta = \{\alpha_1, \alpha_8\}$

The roots $-e_5 - e_6$, $\frac{1}{2}[e_0 + e_1 - e_2 - e_3 - e_4 + e_5 + e_6 - e_7], e_2 - e_1, e_3 - e_2$ form the basis of a $A_4$ root system.

Remark 3.10. It is clear that in the context where $\Theta$ contains $\alpha_1$, adding a root of the form $e_{i+1} + e_i$ in the basis obtained in the projection so that it is attached to $e_{i+2} - e_{i+1}$ next to $e_{i+1} - e_i$ in the Dynkin diagram corresponding to this basis is not possible since both $e_{i+1} + e_i$ and $e_{i+1} - e_i$ cannot be orthogonal to $\alpha_1$. Therefore, it is not possible to obtain a root system of type $D_n$ in the projection.

$\Theta = \{\alpha_1, \alpha_2, \alpha_3\}$

The roots $\alpha_i$ for $i \in \{5, \ldots, 8\}$ form a basis of type $A_4$. Let us add a root of the form $\frac{1}{2}[\pm e_0 \pm e_1 \pm e_2 \cdots \pm e_6 \pm e_7]$ to this basis. Let us assume this root has scalar product with $\alpha_8$ equals to -1, therefore $e_0$ and $e_7$ get a + sign. Since this root is orthogonal to $\alpha_i$ for $i \in \{5, 6, 7\}$, it forces a + sign on $\{e_4, e_5, e_6\}$ too. Further this root has to be orthogonal to $\alpha_1$ and therefore it is : $\frac{1}{2}[e_0 + e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7]$.

$\Theta = \{\alpha_1, \alpha_3, \alpha_4\}$

As in the previous point, the roots $\alpha_i$ for $i \in \{6, 7, 8\}$ and $\frac{1}{2}[e_0 + e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7]$ form the basis of a root system of type $A_4$.

$\Theta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$

As in the previous point, the roots $\alpha_i$ for $i \in \{6, 7, 8\}$ and $\frac{1}{2}[e_0 + e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7]$ form the basis of a root system of type $A_4$. It is immediate to notice the impossibility to complete this root system to a $D_4$.

$\Theta = \{\alpha_6, \alpha_7, \alpha_8\}$

A basis of an $A_4$ root system is given by $\alpha_i$ for $i \in \{2, 3, 4\}$ and $\alpha_1$; to add a root of the form $\frac{1}{2}[\pm e_0 \pm e_1 \pm e_2 \cdots \pm e_6 \pm e_7]$, the latter has to be orthogonal to $\alpha_i$ for $i \in \{2, 3, 4, 6, 7, 8\}$ and has scalar product with $\alpha_2$ equal -1. Such root does not exists in $E_8$. Let us see if we can complete this basis to obtain a root system of type $D_5$. By the Remark 3.10, it is not possible to add $e_2 + e_1$ since it is not orthogonal to $\alpha_1$, a root in the basis of $D_5$. However, the root $\frac{1}{2}[-e_0 + e_1 + e_2 + e_3 - e_4 + e_5 + e_6 + e_7]$, whose scalar product with $\alpha_4$ is -1 appears as the fifth basis root for $D_5$. The sum of the basis' roots is $e_3 + e_2$ which appears in the projection. Therefore a root system of type $D_5$ is obtained.

$\Theta = \{\alpha_1, \alpha_6, \alpha_7, \alpha_8\}$

A basis of an $A_3$ root system is given by $e_4 + e_1, e_2 - e_1$ and $e_3 - e_2$.

$\Theta = \{\alpha_2, \alpha_3\}$
The roots $\alpha_i = \alpha_i$ for $i \in \{5, \ldots, 8\}$ form a basis of type $A_4$. We can add the root of the form $\beta = \frac{1}{2} [e_0 - e_1 - e_2 - e_3 + e_4 - e_5 - e_6 - e_7]$ to complete this basis and obtain a root system of type $A_5$. However, adding $e_0 - e_7$ or some root of the form $\frac{1}{2} [\pm e_0 \pm e_1 \pm e_2 \cdots \pm e_6 \pm e_7]$ to one extreme of the Dynkin diagram to obtain a $D_6$ is not possible. In the first case, $e_0 - e_7$ is not orthogonal to $\beta$; in the second the desired root should be orthogonal to all $\alpha_i$ in $\{2, 3, 6, 7, 8\}$ and $\beta$ and has scalar product with $\alpha_5$ equals to 1. Such root does not exist in $E_8$.

Case $\Theta = \{\alpha_2, \alpha_3\} \cup \{\alpha_5, \alpha_6, \alpha_7, \alpha_8\}$

$\overline{\alpha_1} = \alpha_1$ and $\overline{\alpha_4} = \frac{e_2 + e_3 + e_4 + e_5}{5} - \frac{e_1 + e_4 + e_6}{3}$ and the squared length of $\overline{\alpha_1}$ is $1/3 + 1/5 = 8/15$. The ratios of lengths of projected roots is not compatible with $G_2$ and neither with any root system of classical type and rank 2.

Case $\Theta = \{\alpha_1, \alpha_6, \alpha_7, \alpha_8\}$

$\overline{\alpha_1} = \alpha_i$, $i \in \{2,3\}$ whose squared length are 2.

$\overline{\alpha_3} = e_3 - \frac{1}{2} [e_2 + e_1]$ whose squared length is $3/2$.

$\overline{\alpha_5} = \frac{1}{4} [e_5 + e_6 + e_7 - e_0] - e_4$ whose squared length is $5/4$.

The ratios of lengths do not allow the occurrence of $F_4$. The root $e_4 + e_1$ constitutes the third basis root of a root system of type $A_3$ together with $\alpha_2$ and $\alpha_3$.

Case $\Theta = \{\alpha_3, \alpha_5\}$

$\overline{\alpha_1} = \frac{1}{2} [e_4 + e_5] - \frac{1}{2} [e_2 + e_3]$ whose squared length is 1.

$\overline{\alpha_1} = \alpha_1$

These form the basis of a root system of type $B_2$, further $\overline{\alpha_1} + \overline{\alpha_4}$ and $2\overline{\alpha_4} + \overline{\alpha_1}$ appear in the projection. We can complete this basis with $e_6 - e_7, e_0 + e_7, -e_1 - e_0$, for instance, to form the basis of a root system $B_5$; since the sums of basis’ roots appear in the projection, we obtain $B_5$ in the projection. If one adds the root $\alpha_1$ to $\Theta$, although one still obtain a projection of root $(\alpha_4)$ of squared norm equal to 1, one cannot complete it to form a root system of type $B$.

A root system of type $A_3$ is also obtained from the basis $\{\overline{\alpha_7}, \overline{\alpha_8}, e_1 - e_0\}$.

A similar result can be obtained if one considers $\Theta = \{\alpha_4, \alpha_6\}$. Then

$\overline{\alpha_5} = \frac{1}{2} [e_6 + e_5] - \frac{1}{2} [e_3 + e_4]$ whose squared length is 1.
An appropriate root $\beta = \frac{1}{2}[\pm e_0 \pm e_1 \pm e_2 + e_3 + e_4 - e_5 \pm e_7]$ is playing the role of $\alpha_1$. This basis of type $B_2$ can be completed to obtain a $B_3$ in the projection.

From the case $\Theta = \{\alpha_1, \alpha_4\}$, one easily deduces the cases where $\Theta$ equals $\{\alpha_1, \alpha_2, \alpha_4\}$ or where we obtain a root system of type $A_3$ from $\overline{\alpha_i} = \alpha_i, i \in \{6, 7, 8\}$.

$\Theta = \{\alpha_3, \alpha_4\}$. The projected roots $\overline{\alpha_i} = \alpha_i, i \in \{6, 7, 8\}$ and the root $e_0 - e_1$ along with the root $\frac{1}{2}[e_0 - e_1 - e_2 - e_3 - e_4 + e_5 + e_6 + e_7]$ form the basis of a root system of type $A_5$.

Similarly for the case where $\Theta$ is $\{\alpha_2, \alpha_3, \alpha_4\}$. The projected roots $\overline{\alpha_i} = \alpha_i, i \in \{6, 7, 8\}$ along with the root $\frac{1}{2}[e_0 - e_1 - e_2 - e_3 - e_4 + e_5 + e_6 + e_7]$ form the basis of a root system of type $A_4$.

In the case of $\Theta = \{\alpha_2, \alpha_5\}$ (resp. $\Theta = \{\alpha_3, \alpha_4, \alpha_5\}$) the $\pm e_i \pm e_j$ with $i, j \in \{0, 3, 6, 7\}$ (resp. with $i, j \in \{0, 1, 6, 7\}$) form a $D_4$. If $\Theta = \{\alpha_1, \alpha_2, \alpha_5\}$, this observation is no longer valid. We obtain a $A_4$ in the projection with, for instance, the basis $\alpha_7, \alpha_8, e_0 - e_3$. Idem for $\Theta = \{\alpha_1, \alpha_3, \alpha_5\}$, with the basis $\alpha_7, \alpha_8, e_0 - e_1$.

Let us consider the case of $\Theta = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5\}$. The roots $\overline{\alpha_7}$ and $\overline{\alpha_8}$ have squared norm equal to $5/4$. A root system of type $A_3$ occurs with basis $\alpha_7, \alpha_8$ and $e_0 - e_1$. Adding a root of the form $\frac{1}{2}[\pm e_0 \pm e_1 \pm e_2 \pm e_5 \pm e_6 \pm e_7]$ to this basis is not possible since its scalar product with $e_0 - e_1$ or $\alpha_7$ shall be $-1$, while it is orthogonal to all $\alpha_i$ in $\Theta$; such root with an even number of negative signs does not exist.

$$\Theta = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_8\}$$

$$e_2 = e_3 ; e_5 = e_4, e_6 = -e_7$$

The roots $\pm \frac{1}{2}[e_0 - e_7 + e_6 - e_1]$ have squared norms equal to $1$. It is possible to add $\pm(e_6 - e_1)$ or $\pm(e_7 - e_0)$ to get a basis of $B_2$. However, it is not possible to add a root of the form $\frac{1}{2}[\pm e_0 \pm e_1 \pm e_2 \pm e_5 \pm e_6 \pm e_7]$ of squared norm $2$ since the couples $e_2, e_3$ and $e_4, e_5$ need to have the same sign, and it has to be orthogonal to $\alpha_1$; and this is incompatible with the requirement of having scalar product equal $-1$ with (resp. being orthogonal to) $\pm \frac{1}{2}[e_0 - e_7 + e_6 - e_1]$ or $\pm(e_6 - e_1)$ (resp. $\pm(e_7 - e_0)$).
The squared lengths of projected roots are calculated to determine $C$ and $R$. The root system of highest rank obtained (of rank $\geq 2$) is as follows:

<table>
<thead>
<tr>
<th>$\Theta = {..}$</th>
<th>Squared lengths of projected roots</th>
<th>Chosen roots to calculate $C$ and $R$</th>
<th>Root system of highest rank obtained (of rank $\geq 2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_8$ or any (simple) root of $E_8$</td>
<td>2 and 3/2</td>
<td>$E_7$ and therefore $D_7, A_7$</td>
<td></td>
</tr>
<tr>
<td>${\alpha_1, \alpha_4}$ or ${\alpha_1, \alpha_2, \alpha_4}$ or ${\alpha_1, \alpha_2, \alpha_3, \alpha_4}$</td>
<td></td>
<td>$A_4$</td>
<td></td>
</tr>
<tr>
<td>$\alpha_1, \alpha_5$</td>
<td></td>
<td>$A_5$</td>
<td></td>
</tr>
<tr>
<td>$\alpha_1, \alpha_5, \alpha_2$</td>
<td></td>
<td>$A_4$</td>
<td></td>
</tr>
<tr>
<td>${\alpha_1, \alpha_5, \alpha_3}$ or ${\alpha_1, \alpha_3, \alpha_4, \alpha_5}$</td>
<td></td>
<td>$A_3$</td>
<td></td>
</tr>
<tr>
<td>$\alpha_2, \alpha_5$ and $\alpha_6, \alpha_7$ or $\alpha_8$ or any combination of those three</td>
<td></td>
<td>$A_3$</td>
<td></td>
</tr>
<tr>
<td>${\alpha_1, \alpha_2}$ or ${\alpha_1, \alpha_2, \alpha_3}$</td>
<td></td>
<td>$A_5$</td>
<td></td>
</tr>
<tr>
<td>$\alpha_1, \alpha_3$</td>
<td></td>
<td>$A_5$</td>
<td></td>
</tr>
<tr>
<td>$\alpha_5, \alpha_6, \alpha_7, \alpha_8$</td>
<td></td>
<td>$A_3$</td>
<td></td>
</tr>
<tr>
<td>${\alpha_1, \alpha_3} \cup {\alpha_5, \alpha_6, \alpha_7, \alpha_8}$</td>
<td>$2$ and $7/10$</td>
<td>None</td>
<td></td>
</tr>
<tr>
<td>$\alpha_2, \alpha_3, \alpha_5$</td>
<td>$1, 2, 3/2$</td>
<td>$C=2, R=2, \alpha_1, \alpha_4$</td>
<td>$A_3$</td>
</tr>
<tr>
<td>$\alpha_3, \alpha_5$</td>
<td>$1, 2, 3/2$</td>
<td>$C=2, R=2$</td>
<td>$A_3$</td>
</tr>
<tr>
<td>${\alpha_3, \alpha_4, \alpha_5}$ or ${\alpha_2, \alpha_5}$</td>
<td></td>
<td>$D_4$</td>
<td></td>
</tr>
<tr>
<td>$\alpha_1, \alpha_6, \alpha_7, \alpha_8$</td>
<td>$3/2$ and $2$</td>
<td></td>
<td>$A_3$</td>
</tr>
<tr>
<td>$\alpha_3, \alpha_4$</td>
<td></td>
<td>$A_5$</td>
<td></td>
</tr>
<tr>
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<td></td>
<td>$A_4$</td>
<td></td>
</tr>
<tr>
<td>$\alpha_3, \alpha_4, \alpha_5$</td>
<td></td>
<td>$A_4$</td>
<td></td>
</tr>
<tr>
<td>$\alpha_2, \alpha_5, \alpha_6$</td>
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<td>$A_4$</td>
<td></td>
</tr>
<tr>
<td>${\alpha_2, \alpha_4}$ or ${\alpha_2, \alpha_4, \alpha_1}$ or ${\alpha_2, \alpha_4, \alpha_3, \alpha_1}$</td>
<td></td>
<td>$A_4$</td>
<td></td>
</tr>
<tr>
<td>${\alpha_1, \alpha_8}$ or ${\alpha_1, \alpha_6}$</td>
<td></td>
<td>$A_4$</td>
<td></td>
</tr>
<tr>
<td>$\alpha_1, \alpha_7$</td>
<td></td>
<td>$A_5$</td>
<td></td>
</tr>
<tr>
<td>$\alpha_2, \alpha_3, \alpha_4, \alpha_5$</td>
<td></td>
<td>$A_3$</td>
<td></td>
</tr>
<tr>
<td>$\alpha_6, \alpha_7, \alpha_8$</td>
<td></td>
<td>$D_5$</td>
<td></td>
</tr>
<tr>
<td>$\alpha_1, \alpha_3, \alpha_5, \alpha_8$</td>
<td></td>
<td>$B_2$</td>
<td></td>
</tr>
</tbody>
</table>

Table 4: Roots system occurring in $\Sigma_{E_8}$ for $\Sigma$ of type $E_8$

Root systems of type $D$ occurring, in particular when $\Theta$ contains only one root, are not written systematically.
References


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