Symplectic Models for Unitary groups

A joint work with Dipendra Prasad

Definitions

Let W_i be a symplectic vector space of dimension $2i$ over F. Given a symplectic vector space over F, we have a skew-hermitian space $W_E = W \otimes E$ over E which can be used to define a unitary group $\mathrm{U}(W_E)$ with $\mathrm{Sp}(W) \subset \mathrm{U}(W_E)$.

[Klingen parabolic]

For $G = \mathrm{Sp}(W)$ (or $\mathrm{U}(W_E)$), the Klingen parabolic subgroup Q (resp. P) is the stabilizer of an isotropic line $\langle w \rangle$ in W (resp. W_E). Since any two isotropic vectors in W (or W_E) are conjugate under $\mathrm{Sp}(W)$ (or $\mathrm{U}(W_E)$), the Klingen parabolic subgroups are unique up to conjugacy.

[Klingen mirabolic]

The subgroup Q^1 of Q (resp. P^1 of P) stabilizing the isotropic vector w itself.

Unipotent radical of P_n^1:

$$\mathrm{N}_n(G) = \left\{ \begin{pmatrix} 1 & x_{2n-1} & x_{2n-2} & \cdots & x_2 & z \\ 0 & 1 & 0 & & 0 & y_2 \\ 0 & 0 & 1 & \cdots & 0 & y_3 \\ 0 & & & \ddots & & \vdots \\ 0 & & & \cdots & 1 & y_{2n-1} \\ 0 & \cdots & & 0 & 0 & 1 \end{pmatrix}, \begin{array}{l} x_i\ ,\ \ y_i\ \in E,\ z \in F \\ x_i = \ \bar{y}_i,\ 2 \leq i \leq n-1, \\ x_i = -\bar{y}_i, n \leq i \leq 2n-1 \end{array} \right\}$$

Unipotent radical of Q_n^1:

$$\mathrm{N}_n(S) = \left\{ \begin{pmatrix} 1 & x_{2n-1} & x_{2n-2} & \cdots & x_2 & z \\ 0 & 1 & 0 & & 0 & y_2 \\ 0 & 0 & 1 & \cdots & 0 & y_3 \\ 0 & & & \ddots & & \vdots \\ 0 & & & \cdots & 1 & y_{2n-1} \\ 0 & \cdots & & 0 & 0 & 1 \end{pmatrix}, \begin{array}{l} x_i, y_i, z \in F \\ x_i = \ y_i,\ 2 \leq i \leq n-1, \\ x_i = -y_i, n \leq i \leq 2n-1 \end{array} \right\}$$

Exact sequences:

$$1 \longrightarrow F \longrightarrow N_n(G) \longrightarrow F^{4n-4} \longrightarrow 1$$

$$1 \longrightarrow F \longrightarrow N_n(S) \longrightarrow F^{2n-2} \longrightarrow 1$$

The character μ

Fix a non-trivial character of F, ψ. Assuming $E = F(\sqrt{d})$, $d \in F^\times$, ψ_d character on trace zero elements of E defined by $\psi_d(e) = \psi(\sqrt{d}e)$. Then ψ_n is the character on $N_n(G)$ defined by

$$\psi_n \begin{pmatrix} 1 & x_{2n-1} & x_{2n-2} & \cdots & x_2 & z \\ 0 & 1 & 0 & & 0 & y_2 \\ 0 & 0 & 1 & \cdots & 0 & y_3 \\ 0 & & & \ddots & & \vdots \\ 0 & & & \cdots & 1 & y_{2n-1} \\ 0 & \cdots & & 0 & 0 & 1 \end{pmatrix} = \psi_d(x_{2n-1} + y_{2n-1}) = \psi(\sqrt{d}[x_{2n-1} + y_{2n-1}])$$

Since $x_{2n-1} = -y_{2n-1}$ for elements in $N_n(S)$, the character ψ_n is trivial on $N_n(S)$. Then define a character $\mu : N_n(G) \longrightarrow \mathbb{C}^\times$ which is either ψ_n or trivial.

Local Results

Proposition:
Let π be a smooth representation of the Klingen mirabolic subgroup P_n^1 of $\mathrm{U}(W_n \otimes E)$ which is distinguished by the Klingen mirabolic subgroup $Q_n^1 \subset \mathrm{Sp}(W_n)$, and for the unipotent radical $N_n(G)$ of P_n^1, let π_μ be the maximal quotient of π on which $N_n(G)$ acts by μ. Then π_μ is a smooth representation of the Klingen mirabolic subgroup P_{n-1}^1 of $\mathrm{U}(W_{n-1} \otimes E)$ which is distinguished by the Klingen mirabolic subgroup $Q_{n-1}^1 \subset \mathrm{Sp}(W_{n-1})$.

Corollary:
A smooth representation π of the Klingen mirabolic subgroup P_n^1 of $\mathrm{U}(W_n \otimes E)$ which is distinguished by the Klingen mirabolic subgroup Q_n^1 of the symplectic subgroup $\mathrm{Sp}(W_n)$ carries a nonzero μ_n-linear form for the group of the upper-triangular unipotent matrices in $\mathrm{U}(W_n \otimes E)$ for μ_n given by:

$$\mu_n \begin{pmatrix} 1 & x_1 & * & \cdots & * & * \\ 0 & 1 & x_2 & & * & * \\ 0 & 0 & 1 & x_3 & \cdots & * \\ 0 & & & \ddots & & \vdots \\ 0 & & & \cdots & 1 & x_{2n-1} \\ 0 & \cdots & & 0 & 0 & 1 \end{pmatrix} = \psi_d(\epsilon_1[x_1 + x_{2n-1}] + \epsilon_2[x_2 + x_{2n-2}] + \ldots + \epsilon_{n-1}[x_{n-1} + x_{n+1}])$$

where the ϵ_i are either 0 or 1. Note that the term x_n is missing on the right.

Corollary:
Any representation of $\mathrm{U}(n,n)(F)$ distinguished by $\mathrm{Sp}_{2n}(F)$ is a sub-quotient of a principal series representation of $\mathrm{U}(n,n)(F)$ induced from the Siegel parabolic (with Levi $\mathrm{GL}_n(E)$). In particular, a representation of $\mathrm{U}(n,n)(F)$ distinguished by $\mathrm{Sp}_{2n}(F)$ cannot be cuspidal.

The same is expected for *square-integrable* and even *tempered* representations, indeed :

Conjecture

For F a local field, let $\{\pi\}$ be an L-packet of irreducible admissible representations of $\mathrm{U}(n,n)(F)$ *which we assume to be the L-packet associated to an Arthur packet on* $\mathrm{U}(n,n)(F)$. *Then some member of the set $\{\pi\}$ is distinguished by* $\mathrm{Sp}_{2n}(F)$ *if and only if under basechange, the representation $BC(\pi)$ of* $\mathrm{GL}_{2n}(E)$ *is distinguished by* $\mathrm{Sp}_{2n}(E)$.

Remark: Given the classification of representations of $\mathrm{GL}_{2n}(E)$ which are distinguished by $\mathrm{Sp}_{2n}(E)$ (using Offen-Sayag and Gan-Gross-Prasad) a consequence of the above conjecture is that there should be no tempered representations of $\mathrm{U}(n,n)(F)$ which are distinguished by $\mathrm{Sp}_{2n}(F)$.

A Global Analogue

Let K be a quadratic extension of a number field k.

Theorem Let Π be a cuspidal automorphic representation of $\mathrm{U}(W_n \otimes K)$. Then the period integral of functions in Π on the Klingen mirabolic subgroup Q_n^1 of the symplectic subgroup $\mathrm{Sp}(W_n)$, as well as on the symplectic subgroup $\mathrm{Sp}(W_n)$ is identically zero.

Degenerate Whittaker model for $\mathrm{GL}_n(F)$

In ***Induced representations of reductive p-adic groups II***, Zelevinsky defines a character θ on the group U of upper triangular unipotent elements of $\mathrm{GL}_n(F)$ by

$$\theta(u_{ij}) = \psi(\sum u_{i,i+1}),$$

where $\sum$ runs over all integers $1, 2, \cdots, n-1$ except,

$$n - \lambda_1, n - \lambda_1 - \lambda_2, \cdots, n - \lambda_1 - \lambda_2 - \cdots - \lambda_{k-1},$$

where the integers λ_i are inductively defined with λ_1 being the highest nonzero derivative of π, λ_2 the highest nonzero derivative of π^{λ_1}, and so on.
It is a theorem of Zelevinsky (Corollary 8.3) that there is a linear form $\ell : \pi \longrightarrow \mathbb{C}$ on which the group U of upper triangular unipotent matrices acts by the character θ, and the space of such linear forms has dimension 1.

Conjecture

Let π be an irreducible admissible representation of $\mathrm{GL}(W_n)$ which is distinguished by $\mathrm{Sp}(W_n)$. Write π restricted to $\mathrm{SL}(W_n)$ as a sum of irreducible representations $\pi = \sum \pi_\alpha$ (with multiplicity 1). Then exactly one of the representations π_α is distinguished by $\mathrm{Sp}(W_n)$, and the one which is distinguished by $\mathrm{Sp}(W_n)$ is the one which carries the invariant linear form θ of Zelevinsky defined above.
(There is a unique representation of $\mathrm{SL}(W_n)$ carrying the invariant linear form θ by the multiplicity one assertion of Zelevinsky for the group $\mathrm{GL}_n(F)$.)

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