Symplectic Models for Unitary groups
A joint work with Dipendra Prasad

Definitions

Let $W_i$ be a symplectic vector space of dimension $2i$ over $F$. Given a symplectic vector space over $F$, we have a skew-hermitian space $W = W \otimes E$ over $E$ which can be used to define a unitary group $U(W_i)$ with $(Sp(W_i))^2 \subset U(W_i)$.

[Klingen parabolic]

For $G = Sp(W)$ (or $U(W)$), the Kleinparabolic subgroup $Q$ (resp. $P$) is the stabilizer of an isotropic line $\langle w \rangle$ in $W$ (resp. $W_2$). Since any two isotropic vectors in $W$ (or $W_2$) are conjugate under $Sp(W)$ (or $U(W_2))$, the Kleinparabolic subgroups are uniquely conjugate.

[Klingen mirabolic]

The subgroup $Q_1$ of $Q$ (resp. $P^1$ of $P$) stabilizing the isotropic vector $w$ itself. Unipotent radical of $P^1_n$:

$N_{P^1_n}(G) = \begin{pmatrix}
1 & x_{2n-1} & \cdots & x_{n-2} & x_n & 2 \\
0 & 0 & \cdots & 0 & 1 & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 \\
0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0
\end{pmatrix}$

Unipotent radical of $Q_1$:

$N_{Q_1_n}(S) = \begin{pmatrix}
1 & x_{2n-1} & \cdots & x_{n-2} & x_n & 2 \\
0 & 0 & \cdots & 0 & 1 & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 \\
0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0
\end{pmatrix}$

Exact sequences:

$1 \rightarrow F \rightarrow N_{Q_1_n}(G) \rightarrow F^{2n-4} \rightarrow 1$

$1 \rightarrow F \rightarrow N_{Q_1_n}(S) \rightarrow F^{2n-2} \rightarrow 1$

The character $\mu$

Fix a non-trivial character of $F$, $\psi$. Assuming $E = F(\sqrt{a})$, $d \in F^*$, $\psi_d$ character on trace zero elements of $E$ defined by $\psi_d(x) = \psi(\sqrt{a}x)$. Then $\psi_d$ is the character on $N_{Q_1}(G)$ defined by

$\begin{pmatrix}
1 & x_{2n-1} & \cdots & x_{n-2} & x_n & 2 \\
0 & 0 & \cdots & 0 & 1 & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 \\
0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0
\end{pmatrix} = \psi_d(x_{2n-1} + y_{2n-1})$

Since $x_{2n-1} = -y_{2n-1}$ for elements in $N_{Q_1}(S)$, the character $\psi_d$ is trivial on $N_{Q_1}(S)$. Then define a character $\mu : N_{Q_1}(G) \rightarrow \mathbb{C}$ which is either $\psi_d$ or trivial.

Local Results

Proposition:

Let $\pi$ be a smooth representation of the Kleinparabolic subgroup $P^1_n$ of $U(W_n \otimes E)$ which is distinguished by the Kleinparabolic subgroup $Q^1_n \subset Sp(W_n)$, and for the unipotent radical $N_{Q_1}(G)$ of $P^1_n$, let $\pi_\mu$ be the maximal quotient of $\pi$ on which $N_{Q_1}(G)$ acts by $\mu$. Then $\pi_\mu$ is a smooth representation of the Kleinparabolic subgroup $P^1_n$ of $U(W_n \otimes E)$ which is distinguished by the Kleinparabolic subgroup $Q^1_n \subset Sp(W_n-1)$. Corollary:

A smooth representation $\pi$ of the Kleinparabolic subgroup $P^1_n$ of $U(W_n \otimes E)$ which is distinguished by the Kleinparabolic subgroup $Q^1_n$ of the symplectic subgroup $Sp(W_n)$, as well as on the symplectic subgroup $Sp(W_n)$, is identically zero.

The same is expected for square-integrable and even tempered representations, indeed:

Conjecture

For $F$ a local field, let $\{ \pi \}$ be an L-packet of irreducible admissible representations of $U(n, n)(F)$ which we assume to be the L-packet associated to an Arthur packet on $U(n, n)(F)$. Then some member of the set $\{ \pi \}$ is distinguished by $Sp_{2n}(F)$ if and only if under basechange, the representation $BC(\pi)$ of $GL_{2n}(E)$ is distinguished by $Sp_{2n}(E)$.

Remark:

Given the classification of representations of $GL_{2n}(E)$ which are distinguished by $Sp_{2n}(E)$ (using Offen-Sayag and Gan-Gross-Prasad) a consequence of the above conjecture is that there should be no tempered representations of $U(n, n)(F)$ which are distinguished by $Sp_{2n}(F)$.

A Global Analogue

Let $K$ be a quadratic extension of a number field $k$.

Theorem

Let $\Pi$ be a cuspidal automorphic representation of $U(W_n \otimes K)$. Then the period integral of functions in $\Pi$ on the Kleinparabolic subgroup $Q^1_n$ of the symplectic subgroup $Sp(W_n)$ is zero.

Degenerate Whittaker model for $GL_2(F)$

In Induced representations of reductive $p$-adic groups II, Zelevinsky defines a character $\theta$ on the group $U$ of upper triangular unipotent elements of $GL_2(F)$ by

$\theta(u) = \psi(\sum_i u_{i,i+1})$,

where $\sum$ runs over all integers $1, 2, \ldots, n - 1$ except,

$n - \lambda_1, n - \lambda_2, \ldots, n - \lambda_1 - \lambda_2 - \cdots - \lambda_{k-1}$,

where the integers $\lambda_i$ are inductively defined with $\lambda_1$ being the highest nonzero derivative of $\pi$, $A_2$ the highest nonzero derivative of $\pi^{\lambda_1}$, and so on.

It is a theorem of Zelevinsky (Corollary 8.3) that there is a linear form $\lambda : \mathbb{C} \rightarrow \mathbb{C}$ on which the group $U$ of upper triangular unipotent matrices acts by the character $\theta$, and the space of such linear forms has dimension 1.

Conjecture

Let $\pi$ be an irreducible admissible representation of $GL_n(F)$ which is distinguished by $Sp(W_n)$. Write $\pi$ restricted to $SL(W_n)$ as a sum of irreducible representations $\pi = \sum \pi_{a_i}$ (with multiplicity 1). Then exactly one of the representations $\pi_{a_i}$ is distinguished by $Sp(W_n)$, and the one which is distinguished by $Sp(W_n)$ is the one which carries the invariant linear form $\theta$ of Zelevinsky defined above.

(There is a unique representation of $SL(W_n)$ carrying the invariant linear form $\theta$ by the multiplicity one assertion of Zelevinsky for the group $GL_2(F)$.)

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